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The Inequality of Arithmetic and Geometric Means from Multiple Perspectives

Given three numbers a, b, and c, we can find their mean (or average) as $(a + b + c)/3$. More precisely, this expression yields the *arithmetic mean* of a, b, and c. A different kind of mean, however, uses the product of these numbers instead of their sum. It is called the *geometric mean* and is given by the expression $(abc)^{1/3}$. We may interpret the geometric mean of nonnegative a, b, and c as the side length of a cube whose volume is the same as that of a right rectangular prism with dimensions a, b, and c.

In NCTM's *Focus in High School Mathematics: Reasoning and Sense Making in Algebra* (2010), Graham, Cuoco, and Zimmermann use similar triangles and angles inscribed in circles to prove that, for two nonnegative numbers a and b, the arithmetic mean is greater than or equal to the geometric mean. Symbolically, this is written as $(a + b)/2 \geq (ab)^{1/2}$. This fact, called the inequality of arithmetic and geometric means (AGM), is actually true for any number of nonnegative values. For three nonnegative numbers a, b, and c, for instance, we have $(a + b + c)/3 \geq (abc)^{1/3}$.

Edited by Theodore Hodgson

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On the surface, the AGM inequality seems to be a mere curiosity about the relative magnitudes of the arithmetic and geometric means. As is often the case in mathematics, however, surprising connections can be revealed if we delve deeply. The classic optimization problem—showing that among all rectangles with a fixed perimeter, the square has the largest area—can be expressed through the two-variable AGM. Likewise, the fact that among all right rectangular prisms of a fixed sum of the edge lengths, the cube has the maximum volume can be expressed through the three-variable AGM.

NCTM's Connections Standard recommends that students in grades 9–12 "develop an increased capacity to link mathematical ideas and a deeper understanding of how more than one approach to the same problem can lead to equivalent results, even though the approaches might look quite different" (NCTM 2000, p. 354).

In this article, we embody these recommendations by exploring the AGM inequality in three variables. First, we give an algebraic proof of the AGM inequality using the factorization of a certain symmetric polynomial. Next, we describe geometric inequalities called isoperimetric inequalities and explain how they are related to the AGM inequality. Along the way, we give an elegant proof of the AGM inequality by Cauchy and introduce a beautiful extension called Maclaurin's inequalities. Finally, we give yet another proof of the AGM inequality using Rolle's theorem and calculus. By studying the AGM inequality in three variables and from these different perspectives, both teachers and students can experience and gain appreciation for the interconnected nature of mathematics.

THE AGM INEQUALITY THROUGH ALGEBRA

Consider the equation $x^2 + y^2 - 2xy = (x - y)^2$. Since $(x - y)^2$ is always nonnegative, this factorization implies the inequality $(x^2 + y^2)/2 \geq xy$, with equality if and only if $x = y$. Letting $a = x^2$ and $b = y^2$ so that a and b are nonnegative gives $(a + b)/2 \geq \sqrt{ab}$, with equality if and only if $a = b$. This is the AGM inequality in two variables.

Now consider $x^3 + y^3 + z^3 - 3xyz$, an expression that is a natural analogue to $x^2 + y^2 - 2xy$. The two polynomials share the property that they equal zero when the variables are equal. However, the cubic polynomial does not have an immediate factorization, which its quadratic counterpart does. We observe that $x^3 + y^3 + z^3 - 3xyz$ is *symmetric*, meaning that swapping any pair of variables leaves the polynomial unchanged. When we work with a symmetric expression, it is often useful to "break" the symmetry by replacing it with an equivalent expression that is not symmetric. That can be accomplished here by replacing y by $x + a$ and z by $x + b$, so that the resulting polynomial becomes

$$\begin{aligned} & x^3+(x+a)^3+(x+b)^3-3x(x+a)(x+b) \\ & \quad = 3x(a^2-ab+b^2)+a^3+b^3 \\ & \quad = 3x(a^2-ab+b^2)+(a+b)(a^2-ab+b^2) \\ & \quad = (3x+a+b)(a^2-ab+b^2). \end{aligned}$$

Next, replace a by $y - x$ and b by $z - x$ and combine terms to obtain the desired factorization

$$\text{(1)} \qquad \begin{aligned} & x^3+y^3+z^3-3xyz \\ & \quad = (x+y+z)(x^2+y^2+z^2-xy-xz-yz). \end{aligned}$$

A consequence of this factorization is the AGM inequality in three variables. The right side of (1) contains the polynomial

$$\text{(2)} \qquad \begin{aligned} & x^2+y^2+z^2-xy-xz-yz \\ & \quad = \frac{(x-y)^2+(x-z)^2+(y-z)^2}{2}, \end{aligned}$$

which is zero when $x = y = z$ and otherwise is positive. Moreover, since $x + y + z$ is nonnegative when x, y, and z are nonnegative, we deduce that the right side of (1) is nonnegative, which implies that $(x^3 + y^3 + z^3)/3 \geq xyz$. We then let $a = x^3$, $b = y^3$, and $c = z^3$ to obtain the desired AGM inequality,

$$\frac{a+b+c}{3} \geq (abc)^{1/3},$$

when a, b, and c are nonnegative, with equality if and only if $a = b = c$.

THE AGM INEQUALITY AND GEOMETRY

NCTM's *Focus in High School Mathematics: Reasoning and Sense Making in Algebra* presents a geometric derivation of the AGM inequality in two variables, using the problem of maximizing the area of a rectangle with fixed perimeter (Graham, Cuoco, and Zimmermann 2010). The authors encourage us to extend this geometric investigation to higher dimensions, which we now consider. But first we revisit the AGM inequality in two variables: $(a + b)/2 \geq \sqrt{ab}$ where a and b are nonnegative, with equality if and only if $a = b$. Rearranging this inequality gives $ab/(2a + 2b)^2 \leq 1/16$. Thus, for a rectangle with side lengths a and b, we have

$$\text{(3)} \qquad \frac{\text{area}}{(\text{perimeter})^2} \leq \frac{1}{16},$$

with equality if and only if the rectangle is a square. Among all rectangles of a fixed perimeter, therefore, the square has the maximum area. Here, inequality (3) is an example of an *isoperimetric* ("same perimeter") inequality. This phrase is often used in other settings, where a different measure is used to replace perimeter.

Moving to the third dimension, we consider a right rectangular prism with side lengths a, b, and c. The volume, surface area, and sum of the edge lengths are given by abc, $2(ab + ac + bc)$, and $4(a + b + c)$, respectively. To extend the investigation by Graham, Cuoco, and Zimmermann (2010), we address three isoperimetric questions:

I.1. For a fixed sum of the edge lengths, which prism maximizes the volume?
I.2. For a fixed surface area, which prism maximizes the volume?
I.3. For a fixed sum of the edge lengths, which prism maximizes the surface area?

We show that the prism that resolves all three questions is the cube by proving that the following inequalities are true (for nonnegative a, b, and c), with equality if and only if $a = b = c$:

$$\text{(I.1)} \qquad \frac{(abc)}{\left(4(a+b+c)\right)^3} \leq \frac{1}{12^3}$$

$$\text{(I.2)} \qquad \frac{(abc)^2}{\left(2(ab+ac+bc)\right)^3} \leq \frac{1}{6^3}$$

$$\text{(I.3)} \qquad \frac{2(ab+ac+bc)}{\left(4(a+b+c)\right)^2} \leq \frac{1}{3 \cdot 2^3}$$

As an added bonus, we also derive a more nuanced version of the AGM inequality in three variables.

Proof of the First and Second Isoperimetric Inequalities

We begin by multiplying each side of the inequality by $(4(a+b+c))^3$ to reveal that the first inequality (I.1) is actually equivalent to

$$abc \le \left(\frac{a+b+c}{3}\right)^3 \text{ or } (abc)^{1/3} \le \frac{a+b+c}{3},$$

which is the AGM inequality in three variables. Here, we give a proof of the three-variable AGM by Cauchy (Bradley and Sandifer 2009, p. 306; Cauchy 1821, pp. 457–59).

First, we establish the following relationships for any four variables:

$$(4) \qquad \begin{aligned}(abcd)^{1/4} &= (ab)^{1/4}(cd)^{1/4} \\ &\le \left(\frac{a+b}{2}\right)^{1/2}\left(\frac{c+d}{2}\right)^{1/2} \\ &\le \frac{(a+b)/2+(c+d)/2}{2} \\ &= \frac{a+b+c+d}{4}.\end{aligned}$$

The two inequalities in (4) are derived from the AGM inequality in two variables; we see here the process of reducing the problem to a previously solved one. Next, break the symmetry temporarily by setting $d = (a+b+c)/3$. This gives

$$\begin{aligned}&\frac{a+b+c+d}{4} \\ &\quad= \frac{a+b+c+(a+b+c)/3}{4} \\ &\quad= \frac{a+b+c}{4}+\frac{a+b+c}{12} = \frac{a+b+c}{3}.\end{aligned}$$

Thus, (4) becomes

$$(abc)^{1/4}((a+b+c)/3)^{1/4} \le (a+b+c)/3.$$

This inequality is obviously true for $a = b = c = 0$, so assume that at least one of the variables is positive. Thus, $(abc)^{1/4} \le ((a+b+c)/3)^{3/4}$, which can be simplified to obtain the AGM inequality in three variables.

It can be shown that the second isoperimetric inequality is equivalent to the first inequality, with a, b, and c replaced by their respective reciprocals. In each case, a, b, and c are side lengths of a rectangular prism, so we can assume that they are nonzero.

Proof of the Third Isoperimetric Inequality

Start with the polynomial $a^2 + b^2 + c^2 - ab - ac - bc$, which we recognize from equation (2). Rewrite this expression (using $a^2 = a^2/2 + a^2/2$, and the same replacements for b^2 and c^2) to obtain

$$\begin{aligned}&a^2+b^2+c^2-ab-ac-bc \\ &\quad= \left(\frac{a^2}{2}-ab+\frac{b^2}{2}\right)+\left(\frac{a^2}{2}-ac+\frac{c^2}{2}\right) \\ &\qquad+\left(\frac{b^2}{2}-bc+\frac{c^2}{2}\right) \\ &\quad= \frac{(a-b)^2+(a-c)^2+(b-c)^2}{2}.\end{aligned}$$

This expression is zero when $a = b = c$ and otherwise is positive, so that $a^2 + b^2 + c^2 - ab - ac - bc \ge 0$. This is equivalent to $3(ab+ac+bc) \le (a+b+c)^2$ (as can be shown by expanding $(a+b+c)^2$ and rearranging terms), which can be further rearranged to obtain the third inequality, thus completing the proof.

A More Nuanced Version of the AGM Inequality

Combining the second and third isoperimetric inequalities through some algebraic manipulation yields

$$(abc)^{1/3} \le \left(\frac{ab+ac+bc}{3}\right)^{1/2} \le \frac{a+b+c}{3}$$

and suggests the following extension to four variables:

$$\begin{aligned}(abcd)^{1/4} &\le \left(\frac{abc+abd+acd+bcd}{4}\right)^{1/3} \\ &\le \left(\frac{ab+ac+ad+bc+bd+cd}{6}\right)^{1/2} \\ &\le \frac{a+b+c+d}{4}.\end{aligned}$$

The denominators of these four expressions—namely, 1, 4, 6, and 4—are obtained from the fourth row of Pascal's triangle. This inequality is indeed true, and analogous generalizations called Maclaurin's inequalities exist for higher dimensions (Ben-Ari and Conrad 2014).

THE AGM INEQUALITY THROUGH CALCULUS

In this section, we prove the AGM inequality in three variables using ideas from calculus. As a motivation, we revisit the two-variable case. Let a and b be real numbers and consider

$$f(x) = (x+a)(x+b) = x^2 + (a+b)x + ab.$$

Since $f(x)$ has real roots, its discriminant is nonnegative, with equality if and only if $f(x)$ has a double root. Thus, $(a+b)^2 - 4ab \geq 0$, with equality if and only if $a = b$, from which we obtain the AGM inequality in two variables.

The two-variable case suggests that we try something similar with three variables. Let a, b, and c be real numbers and consider

$$\begin{aligned} f(x) &= (x+a)(x+b)(x+c) \\ &= x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc. \end{aligned}$$

Instead of trying to determine a condition for a cubic polynomial to have real roots (using something like a discriminant), we rely on a familiar theorem from calculus (see **fig. 1**). For

$$f(x) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc,$$

we have

$$f'(x) = 3x^2 + 2(a+b+c)x + (ab+ac+bc).$$

We may assume that a, b, and c are distinct, without loss of generality. Since $f(-a) = f(-b) = f(-c) = 0$, Rolle's theorem implies that $f'(x)$ has two distinct real roots. Note that if $a = b = c$, then the AGM inequality follows trivially. If two of the variables are equal—say, $a = b$—then $f'(-a) = 0$, and Rolle's theorem implies that $f'(x)$ has a root between $-a$ and $-c$, which still gives us two distinct real roots for $f'(x)$. Thus, the discriminant of $f'(x)$ is positive, so

$$4(a+b+c)^2 - 12(ab+ac+bc) > 0.$$

This can be rewritten as

Rolle's theorem: Let $f(x)$ be continuous on $\alpha \leq x \leq \beta$ and differentiable on $\alpha < x < \beta$. If $f(\alpha) = f(\beta) = 0$, then there exists γ with $\alpha < \gamma < \beta$ such that $f'(\gamma) = 0$

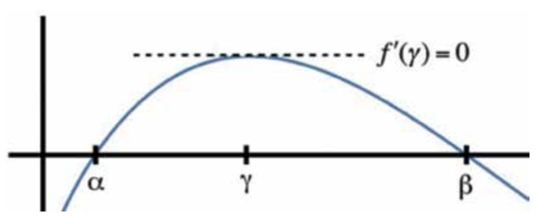

Fig. 1 The graph illustrates Rolle's theorem.

$$(5) \qquad \frac{ab+ac+bc}{3} < \left(\frac{a+b+c}{3}\right)^2,$$

which is equivalent to the third of our isoperimetric inequalities (I.3 from the previous section). Note how differentiation reduced the cubic to a quadratic, which we already knew how to analyze.

Last, we give a calculus-based proof of the AGM inequality in three variables. Consider the cubic $f(x)$ above and define $g(y)$ by replacing x by $1/y$ and multiplying by y^3:

$$\begin{aligned} g(y) &= y^3 \bullet f(1/y) \\ &= abc \bullet y^3 + (ab+ac+bc)y^2 + (a+b+c)y + 1. \end{aligned}$$

The roots of $g(y)$ are $-1/a$, $-1/b$, and $-1/c$ (where we again assume that a, b, and c are nonzero and distinct). As before, we reduce $g(y)$ to a quadratic by differentiating:

$$g'(y) = 3abc \bullet y^2 + 2(ab+ac+bc)y + (a+b+c).$$

By Rolle's theorem, the roots of $g'(y)$ are real and distinct. Thus, its discriminant is positive so that

$$4(ab+ac+bc)^2 - 12abc(a+b+c) > 0.$$

This can be rewritten as

$$(6) \qquad abc\left(\frac{a+b+c}{3}\right) < \left(\frac{ab+ac+bc}{3}\right)^2.$$

Combining (5) and (6), we get

$$abc\left(\frac{a+b+c}{3}\right) < \left(\frac{a+b+c}{3}\right)^4,$$

which simplifies to

$$abc < \left(\frac{a+b+c}{3}\right)^3$$

—the AGM inequality in three variables.

In a typical calculus course, Rolle's theorem is treated solely as a lemma to the mean value theorem. The proofs here illustrate one of the alternative and interesting ways in which Rolle's theorem may be used, and we recommend that calculus teachers include these ideas in their course.

RECURRING THEMES

Several underlying themes run throughout this article. The first is the mathematical practice of reducing a problem to an easier or previously solved one, as suggested by Pólya (1962) in his problem-solving heuristic: "It may happen that our original problem involves concepts with which we are not used to dealing. In such a situation, it may be advisable to try some easier problem involving the same concepts that would thus become a (rather remote) auxiliary problem to our original problem" (p. 43). We saw this in the derivation of (4) in Cauchy's proof of the AGM inequality as well as in the use of differentiation to reduce cubic polynomials into quadratics that we knew how to analyze.

The second theme is symmetry. We saw not only the value of symmetry in polynomials but also the potential value of breaking algebraic symmetry—as we did to derive the factorization in (1) and Cauchy's proof.

The third theme is the history of mathematics, which we saw in Cauchy's proof and the extension to Maclaurin's inequalities. Interested teachers and students might further pursue this history by reading Maclaurin's original manuscript, in which he first states his inequality (Maclaurin 1729). It is also a good example of how mathematics was written in an earlier time.

It is important to note that, when using this material with students, reading this material (or watching it written on the board) may feel very much like symbol pushing, without much meaning. A lot of algebraic calculation is happening in this work. Going through the calculations for oneself is another matter entirely—structure is revealed in carrying out the calculations. Grappling with these ideas gives students an opportunity to connect symbolic operations with real meaning, particularly as they see the ideas recurring in different mathematical contexts.

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