The Common Core K–12 Mathematics Standards

This document provides grade level standards for mathematics in grades K–8, and high school standards organized under the headings of the *College and Career Readiness Standards in Mathematics*. Students reaching the readiness level described in that document (adjusted in response to feedback) will be prepared for non-remedial college mathematics courses and for training programs for career-level jobs. Recognizing that most students and parents have higher aspirations, and that ready for college is not the same as ready for mathematics-intensive majors and careers, we have included in this document standards going beyond the readiness level. Most students will cover these additional standards. Students who want the option of entering STEM fields will reach the readiness level by grade 10 or 11 and take precalculus or calculus before graduating from high school. Other students will go beyond readiness through statistics to college. Other pathways can be designed and available as long as they include the readiness level. The final draft of the K–12 standards will indicate which concepts and skills are needed to reach the readiness level and which go beyond. We welcome feedback from states on where that line should be drawn.

English Language Learners in Mathematics Classrooms

English language learners (ELLs) must be held to the same high standards expected of students who are already proficient in English. However, because these students are acquiring English language proficiency and content area knowledge concurrently, some students will require additional time and all will require appropriate instructional support and aligned assessments.

ELLs are a heterogeneous group with differences in ethnic background, first language, socio-economic status, quality of prior schooling, and levels of English language proficiency. Effectively educating these students requires adjusting instruction and assessment in ways that consider these factors. For example ELLs who are literate in a first language that shares cognates with English can apply first-language vocabulary knowledge when reading in English; likewise ELLs with high levels of schooling can bring to bear conceptual knowledge developed in their first language when reading in a second language. On the other hand, ELLs with limited or interrupted schooling will need to acquire background knowledge prerequisite to educational tasks at hand. As they become acculturated to US schools, ELLs who are newcomers will need sufficiently scaffolded instruction and assessments to make sense of content delivered in a second language and display this content knowledge.

While some ELLs are economically and educationally advantaged, this is not the case for many of these students. Moreover, once in the U.S., the majority of ELLs attend high poverty schools with high percentages of other ELLs. These schools often lack the resources and capacity needed to help ELLs reach high academic standards. However, schools and districts can be assisted in providing a positive learning environment that capitalizes on the linguistic and cultural diversity of the student body.

Language proficiency is a complex construct that can reflect proficiency in multiple contexts, modes, and academic disciplines. Current measures of language proficiency may not give an accurate picture of an individual's language competence. In particular, we do not have measures or assessments for language proficiency related to competence in mathematics for different ages or mathematical topics. These two facts can confuse discussions of mathematics instruction for ELLs. In particular, because of the complexity of language proficiency and the limitations of the label "English Language Learner" as currently implemented, instructional decisions should not be made solely based on that label. However, research on language and mathematics education for this student population does provide a few clear results to guide practices for teaching ELLs mathematics:

- English learners can participate in mathematical discussions as they learn English (Moschkovich, 1999a, 2002, 2007a, 2007b, 2007d).

- Mathematics instruction for students who are learning English should draw on multiple resources and modes available in classrooms—such as objects, drawings, inscriptions, and gestures—as well as home languages and mathematical experiences outside of school.

- While mathematics instruction for ELLs should address mathematical discourse and academic language, this involves much more than vocabulary instruction.

Basic principles for improving the mathematics achievement of ELLs

Language is a resource for learning mathematics, it is not only a tool for communicating, but also a tool for thinking and reasoning mathematically. All languages (English, Spanish, Tagalog,etc.) and language varieties (different dialects, home or everyday ways of talking, vernacular, slang, etc.) provide resources for mathematical thinking, reasoning, and communicating.

Regular and active participation in the classroom—not only reading and listening but also discussing, explaining, writing, representing, and presenting—is crucial to ELLs' success in mathematics, and that ELLS can produce explanations, presentations, etc. and participate in classroom discussions *as they are learning English* (Moschkovich, 1999 and 2007).

- ELLs, like English-speaking students, require regular access to teaching practices that are most effective for improving student achievement. These practices include: a) Keeping mathematical tasks at high-cognitive demand (Henningsen & Stein, 1997; Silver & Stein, 1996); b) teachers and students attend explicitly to concepts (Hiebert & Grouws, 2007), and c) students wrestle with important mathematics (Hiebert & Grouws, 2007).
- See the evidence of ELLs' mathematical thinking, hear how ELLs use language to communicate about mathematics, understand the competence that ELLS bring to the classroom, build on this competence, and provide access to opportunities for advancing their mathematics learning.

Overall, research suggests that:

- Classroom instruction should allow bilingual students to choose the language they prefer for arithmetic computation. Language switching can be swift, highly automatic, and facilitate rather than inhibit solving word problems in the second language, as long as the student's language proficiency is sufficient for understanding the text of the word problem.
- Instruction should ensure that students understand the text of word problems before they attempt to solve them.
- Instruction should include a focus on "mathematical discourse" and "academic language" because these are important for English learners. Although it is crucial that students who are learning English have opportunities to communicate mathematically, this is not primarily a matter of learning vocabulary. Students learn to participate in mathematical reasoning, not by learning vocabulary, but by making conjectures, presenting explanations, and/or constructing arguments.
- While vocabulary instruction is important, it is not sufficient for supporting mathematical communication. Furthermore, vocabulary drill and practice are not the most effective instructional practices for learning vocabulary. Instead, research has demonstrated that vocabulary learning occurs most successfully through instructional environments that are language-rich, actively involve students in using language, require that students both understand spoken or written words and also express that understanding orally and in writing, and require students to use words in multiple ways over extended periods of time (Blachowicz, Camille, and Peter Fisher, 2000). To develop written and oral communication skills, students need to participate in negotiating meaning for mathematical situations and in mathematical practices that require output from students (Moschkovich, 2009).

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Access for Students with Disabilities

The Common Core Standards articulate rigorous expectations in the areas of mathematics, reading, writing, and speaking and listening in order to prepare students to be college- and career-ready. These standards identify the knowledge and skills students must acquire in order to be successful. Research shows that students with disabilities are capable of high levels of learning and should not be limited by low expectations and watered down curriculum. It is imperative that these highly capable students—regardless of their disability—are held to the same expectations articulated in the Core Standards as other students.[1]

However, *how* these high standards are taught is of the utmost importance in reaching students with special needs. When learning the knowledge and skills represented in the Core Standards, students with disabilities may need accommodations or—in exceptional cases—modified goals, incorporated in an individualized education program (IEP),[2] to help them access information or demonstrate their knowledge. Students might be precluded from reaching particular standards given the nature of the standard itself. In instances when a standard asks students to perform actions they are physically incapable of, students will need to be presented with alternative options to demonstrate similar knowledge and skills within the range of their abilities. Accommodations based on individual needs allow students of all disability levels to learn within the framework of the Core.

Meeting English Language Arts (ELA) Standards

Reading, writing, speaking, and listening standards often require accommodations for students with disabilities. In the case of students who are deaf, a standard that calls for "listening" should be interpreted to include reading sign language. In a similar vein, "speaking" as it occurs in standards for certain students with speech impairments should be read broadly to include "communication" or "self-expression." Students who are blind or have low vision should be able to read via Braille, screen reader technology, or other assistive technology to demonstrate their comprehension

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1 Research suggests that the vast majority of the population of students with intellectual impairments *can* achieve proficiency when they receive high quality instruction in the grade-level content and appropriate accommodations.

2 According to the Individuals with Disabilities Act (IDEA), an IEP includes appropriate accommodations that are necessary to measure the individual achievement and functional performance of a child.

skills. "Writing" should not preclude the use of a scribe, computer, or speech-to-text technology for students with disabilities that interfere with putting pen to paper. In the case of students with intellectual impairments—less than 2 percent of the total population of all students and less than 20 percent of students with disabilities—accommodations should allow them to demonstrate their knowledge and skills through alternative modes like text to speech software or reading aloud. For these students, writing may involve the use of pictures to assist in illustrating plot or argument, or offering them the opportunity to "choose words and phrases" by selecting from options rather than generating direct answers. With appropriate accommodations and support, students with all levels of disabilities can participate in the general education curriculum and achieve grade-level proficiency with regard to the ELA content and skills articulated in the Core.

Meeting Mathematics Standards

In curriculum for students with disabilities, ELA skills often take precedence over mathematics understanding. However, most of these students can master mathematical concepts with accommodations in instructional delivery and the use of specialized technology, including computers and calculators. For example, students with visual disabilities might require enhanced verbal descriptions from teachers and the use of large-print to demonstrate subsequent knowledge. Students who are deaf might require visual aids such as charts, diagrams, and mental images and increased reliance on computers and calculators. Manipulatives can enable students with intellectual impairments to grasp abstract concepts and continue learning. Evidence suggests that students with disabilities, even those with full intellectual abilities, tend to lag behind their peers in mathematics achievement; strong curriculum that gives equal priority to mathematics and ELA will help these students succeed.

In short, while the standards remain and retain high expectations of students, they may need to be translated and occasionally modified to appropriately apply to students with disabilities, including all levels of intellectual impairment. Every student deserves to be treated with respect, and every student deserves an outstanding education. Promoting a culture of high expectations for all students is a fundamental goal of the Core Standards. Reaching students with disabilities requires broadening our understanding of what the standards say and being ready to make appropriate accommodations and/or modifications to meet individual students' needs.

How to read this document

The K–8 standards are organized by grade level. Within each grade level there are several headings, each one the title of a single progression having significant presence in the grade in question. Under each of these progression headings, there appear **core standards**, divided into standards describing concepts students should understand and standards describing skills students should acquire. A typical progression spans a number of grades, but does not span all of K-8. The progressions and their grade spans are listed at the end of the document.

The high school standards are not organized by grade level or by course, but rather are organized under headings of the *College and Career Ready Standards for Mathematics*: Expressions, Equations, Functions, Coordinates, Modeling, Statistics, Probability, and Geometry.[3] Subheadings under each heading refer either to mathematical practices or to principle topics, and core standards are listed under each subheading, as in the K-8 standards. The subheadings are not necessarily curricular units, but rather can describe concepts and skills that are revisited throughout a student's high school career. This design necessitates a future effort to develop course sequences (either traditional or integrated).

[3] Number and Quantity are not included, since they are principally the domain of K–8. In response to feedback, the headings have been reordered and Shape has been renamed to Geometry.

Mathematical Practice[4]

Proficient students expect mathematics to make sense. They take an active stance in solving mathematical problems. When faced with a non-routine problem, they have the courage to plunge in and try something, and they have the procedural and conceptual tools to carry through. They are experimenters and inventors, and can adapt known strategies to new problems. They think strategically.

Students who engage in these practices discover ideas and gain insights that spur them to pursue mathematics beyond the classroom walls. They learn that effort counts in mathematical achievement.[a] These are practices that expert mathematical thinkers encourage in apprentices. Encouraging these practices in our students should be as much a goal of the mathematics curriculum as is teaching specific content topics and procedures.[b] Taken together with the Standards for Mathematical Content, they support productive entry into college courses or career pathways.

Core Standards · Students can and do:

1 Attend to precision.

Mathematically proficient students organize their own ideas in a way that can be communicated precisely to others, and they analyze and evaluate others' mathematical thinking and strategies noting the assumptions made. They clarify definitions. They state the meaning of the symbols they choose, are careful about specifying units of measure and labeling axes, and express their answers with an appropriate degree of precision. Rather than saying, "let v be speed and let t be time," they would say "let v be the speed in meters per second and let t be the elapsed time in seconds from a given starting time." They recognize that when someone says the population of the United States in June 2008 was 304,059,724, the last few digits indicate unwarranted precision.

2 Construct viable arguments.

Mathematically proficient students understand and use stated assumptions, definitions and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They break things down into cases and can recognize and use counterexamples. They use logic to justify their conclusions, communicate them to others and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose.

3 Make sense of complex problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They consider analogous problems, try special cases and work on simpler forms. They evaluate their progress and change course if necessary. They try putting algebraic expressions into different forms or try changing the viewing window on their calculator to get the information they need. They look for correspondences between equations, verbal descriptions, tables, and graphs. They draw diagrams of relationships, graph data, search for regularity and trends, and construct mathematical models. They check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"

4 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern. For example, in $x^2 + 5x + 6$ they can see the 5 as $2 + 3$ and the 6 as 2×3. They recognize the significance of an existing line in a geometric figure and can add an auxiliary line to make the solution of a problem clear. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects. For example, by seeing $5 - 3(x$

[4] Slated for review and editing, based on feedback to the College and Career Readiness Standards, and in order to apply more naturally to elementary school as well.

$- y)^2$ as 5 minus a positive number times a square, they see that it cannot be more than 5 for any real numbers x and y.[b]

5 Look for and express regularity in repeated reasoning.

Mathematically proficient students pay attention to repeated calculations as they carry them out, and look both for general algorithms and for shortcuts. For example, by paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, they might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel in the expansions of $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ leads to the general formula for the sum of a geometric series. As they work through the solution to a problem, proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.[b]

6 Reason quantitatively.

Quantitative reasoning is a way of thinking by which one reasons with quantities and about relations among quantities. It entails habits of creating a coherent image of the problem at hand; considering the units involved; continually attending to the meaning of quantities, not just how to compute them; and having multiple images of a concept and being flexible in transitioning among them. In problems dealing with quantitative relationships, students exercise two inseparable abilities: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referential meanings for the symbols involved in the manipulation.

7 Make strategic decisions about the use of technological tools.

Mathematically proficient students consider the available tools when solving a mathematical problem, whether pencil and paper, ruler, protractor, graphing calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software. They are familiar enough with all of these tools to make sound decisions about when each might be helpful. They use mathematical understanding and estimation strategically, attending to levels of precision, to ensure appropriate levels of approximation and to detect possible errors. They are able to use these tools to explore and deepen their understanding of concepts.

Mathematics: Kindergarten[5]

Developing Coherent Understanding

[Temporarily removed for editing.]

Counting and Cardinality

Core Standards · Students understand that:

1. The number words have a standard order.
2. In counting, each object receives one and only one number word.
3. The last number word tells the number of objects.
4. Numbers said later in the count refer to larger quantities.
5. Counting on 1 more is the same as adding 1. That is, one more than a number is the next number in the count.

Core Standards · Students can and do:

a. Count by ones from 1 to 100; count by tens to 100.[6]
b. Count forward from a given number within the known sequence (instead of always counting forward from 1).[7]
c. See collections of up to 10 objects as being composed of subgroups.
d. Count to answer "how many?" questions with up to 10 things in various arrangements (e.g., array, circular, scattered), or up to 25 things if in a row.
e. Write numerals from 1 to at least 30.

Base Ten Computation

Core Standards · Students understand that:

1. Ten ones make a tens unit (ten things can be thought of as bundled into a single unit).
2. Decade words refer to groups of tens units. For example, thirty refers to a group of three tens units.
3. A teen number[8] is a ten and some ones. The number 10 can be thought of as a ten and no ones.
4. Any teen number is larger than any single digit number. Teen numbers are ordered according to their ones digits.
5. A two-digit number is some tens and some ones. For example, 29 is two tens and nine ones.

Core Standards · Students can and do:

a. Make 10 with each number from 1 to 9 (i.e., know the number that makes 10 with the given number).
b. Show each teen number as a ten and some ones.

Early Relations and Operations

Core Standards · Students understand that:

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[5] Some material is used verbatim from National Research Council. (2009). *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity*. Committee on Early Childhood Mathematics, Christopher T. Cross, Taniesha A. Woods, and Heidi Schweingruber, Editors. Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: The National Academies Press.
[6] To "count" here means only to say the number words, not to determine how many objects are in a collection.
[7] To "count" here means only to say the number words, not to determine how many objects are in a collection.
[8] Glossary: Teen number. A whole number that is greater than or equal to 11 and less than or equal to 19.

1. Adding is putting two groups together or putting some more with a group, and subtracting is taking some from a group.
2. Addition and subtraction can be represented with physical or mental objects (including fingers), pictures, drawings, sounds (e.g. number words), motions, or equations.
3. Adding can be recorded by an expression, as when "three more than six" is recorded as $6 + 3$, or by an equation that also shows the answer ($6 + 3 = 9$). Likewise, subtracting can be recorded by an expression, as when "how much more than 9 is 5" is recorded as $9 – 5$, or by an equation that also shows the answer ($9 – 5 = 4$).
4. Breaking apart a group can be recorded in an equation such as $8 = 5 + 3$. Breaking apart a group in more than one way can be recorded in an equation such as $7 + 6 = 10 + 3$.
5. In all equations, the equals sign indicates that the values on either side are the same.

Core Standards · Students can and do:

a. Use matching and counting strategies to decide whether one set is more than, less than, or equal to another set in number of objects (less than or equal to 10).
b. Compare and order numbers less than or equal to 10.
c. Use concrete objects to determine the answer to addition and subtraction word problems and additions and subtractions with totals less than or equal to 10.
d. Experience enough problem situations so that additions to five and the corresponding subtractions and some additions and subtractions within ten become well known.

Quantity and Measurement

Core Standards · Students understand that:

1. Things have attributes—such as length, weight, capacity, loudness, softness, and so on. A single thing might have several attributes of interest (as when we focus on a child's height and gender).

Core Standards · Students can and do:

a. Directly compare two objects to see which one has "more of" a shared attribute.
b. Rank three objects by a shared attribute (especially length), and use transitivity[9] to compare two objects indirectly.
c. ♦ Classify objects or people into predetermined categories, and count the numbers in each category. List the categories and counts in order by count. (Each category count less than or equal to 10.)[10]

Shapes

Core Standards · Students understand that:

1. Names refer to shapes regardless of orientation or overall size.[11]

Core Standards · Students can and do:

a. Study a range of 2D and 3D shapes, in different sizes and orientations, and discuss their properties, similarities, and differences using informal language.

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[9] Glossary: Transitive property of measurement order: If one object is bigger than a second, and the second object is bigger than a third object, then the first object is bigger than the third object.
[10] The symbol ♦ indicates material in data analysis and statistics that appears under another progression heading in order to make an important connection.
[11] For example, a square rotated to form a "diamond" is still a square, even though it is rotated. Students at this grade might need to physically rotate a shape until it is "level" before they can correctly name it.

b. Move shapes using translations, reflections and rotations.[12]

[12] This is not meant to be assessed by showing students a picture of a shape and asking them to draw or select a translated, reflected or rotated version of it.

Mathematics: First Grade[13]

Developing Coherent Understanding

[Temporarily removed for editing.]

Early Relations and Operations

Core Standards · Students understand that:

1. Counting on is an efficient method of counting all, in which the initial count of the first addend is omitted.
2. Addition and subtraction apply to situations of joining, separating, part-part-whole, and comparing quantities to one another.[14] These situations can be represented by addition and subtraction equations such as $7 + 5 = 12$, $10 = 5 + 5$, and so on.
3. Addition and subtraction are inverse operations; that is $10 – 8$ can be found by thinking $8 + 2 = 10$.
4. When any two of the numbers in an addition or subtraction equation are known, the unknown number can be found.
5. One-to-one dealing of objects in a collection (e.g., "One for you, one for me, one for him, …") creates fair shares.

Core Standards · Students can and do:

a. Use counting on strategies or decomposing strategies for additions and subtractions within 20.
b. Solve addition problems containing three addends.
c. Use objects, pictures and story contexts to explain what happens when the order of addends in a sum is changed, when 0 is added to a number, and when one addend in a sum is increased by 1 and another decreased by 1.
d. Experience enough problem situations so that many or all sums and differences within 20 become well known.
e. Use drawings and equations to represent and solve word problems involving addition and subtraction.[15]
f. ♦ Organize, represent and interpret data with several categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another
g. Create *n* fair shares from a collection of objects. Identify the size of one share, and recognize the original collection as *n* copies of a single share.

Quantity and Measurement

Core Standards · Students understand that:

1. Lengths can be added by placing long objects, rods, or unit cubes end to end in a straight line. The total length is the same in whatever order the rods are placed.

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[13] Some material is used verbatim from National Research Council. (2009, op. cit.)

[14] In join and separate problems, there is change over time. In part-part-whole problems, two quantities make up a whole in a static situation. Compare problems involve two quantities and the difference between them. Compare problems add specificity to the notions of greater than and less than.

[15] Include join, separate, part-part-whole, and compare problems, with unknowns in all positions. Represent these situations with equations that use a small square or a ? for the unknown.

2. Lengths can be compared by placing rods side by side, with one end lined up. The difference in length is how far the longer extends beyond the end of the shorter.
3. Lengths are measured (assigned numerical values) by comparing them to other lengths—that is, by using another object as a length unit. The length of an object can be expressed numerically by counting the number of length units that span it with no gaps or overlaps.
4. When an object or figure is decomposed into several pieces, the length of the whole can be found by placing the pieces end to end in any order.
5. A sum of two whole numbers represents a total length; a difference of two whole numbers represents a difference in length.
6. Durations of time are measured by comparing them to other durations of time, such as the earth's rotation period, or the time a minute hand takes to complete a circle around a clock face.

Core Standards · Students can and do:

a. Using an object as a length unit, measure, compare and estimate length.[16]
b. Using an object as a length unit, determine total length by adding lengths of two parts.[17] Compare lengths using addition and subtraction.
c. Decompose circles and rectangles into 2 and 4 equal parts. Describe the parts using the words "halves" and "quarters," and using the phrases "half of" and "quarter of." Describe the wholes as twice or four times as large as the parts.
d. Tell time in hours from clocks; subtract to find whole-hour durations on a clock (within AM or within PM).

Base Ten Computation

Core Standards · Students understand that:

1. In comparing two-digit numbers, the number with more tens units is larger; if the number of tens units is the same in each, the number of ones units decides.
2. In adding or subtracting 2-digit numbers, one adds or subtract like units (tens units and tens units, or ones units and ones units).

Core Standards · Students can and do:

a. Count to 100 or beyond, switching appropriately to the new decade after a 9 has been said in the ones place.
b. Compare and order numbers to 100 based on meanings of the tens and ones places.
c. Easily write numerals to 20; write numerals to 100.
d. Use break-apart and make-a-ten strategies to add and subtract with teen totals as in $7 + 6 = 10 + 3$ and $17 – 9 = 17 – 7 – 2$.
e. Find 10 more or 10 less than a number without having to count.
f. Add one-digit numbers to two-digit numbers, and add multiples of 10 to one-digit and two-digit numbers.
g. Represent addition of two-digit numbers using 10-rods and unit cubes,[18] including rearranging rods and cubes to show regrouping when needed.
h. Add two-digit numbers to two-digit numbers using strategies based on place value, Properties of Arithmetic, or the inverse relationship between addition and subtraction.

Shapes

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[16] Select and iterate units, partition into equal parts, and compare lengths indirectly by using a reference length.
[17] Restrict to whole-unit lengths.
[18] Any concrete model that can show individual units and ten connected units will do.

Core Standards · Students understand that:

1. Several shapes can be joined together to form a larger shape. A single shape can also be visualized as a collection of smaller shapes joined together.
2. Decomposing larger shapes into equal-sized parts creates fair shares.
3. When an identical figure is decomposed into more fair shares, the shares are smaller than in the first instance.

Core Standards · Students can and do:

a. Form different 2D figures with cutouts of rectangles, squares, triangles, semicircles, and quarter-circles.[19]
b. Form different 3D figures with concrete models of cubes, rectangular prisms, cones, and cylinders.[20]
c. Decompose 2D shapes into rectangles, squares, triangles, semicircles, and quarter-circles, including decomposing into fair shares.

[19] From Singapore Primary 2
[20] From Singapore Primary 2

Mathematics: Second Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Operations and the Problems They Solve

Core Standards · Students understand that:

1. Addition and subtraction apply to situations of joining, separating, part-part-whole, and comparing quantities to one another.[21] These situations can be represented by addition and subtraction equations such as $17 + 5 = 22$, $36 = 56 – 26$, and so on.
2. Addition and subtraction are inverse operations; that is $100 – 98$ can be found by thinking $98 + 2 = 100$.
3. Numbers can be added and subtracted only when they refer to the same underlying unit.

Core Standards · Students can and do:

a. Use representations (objects, pictures, story contexts) to describe and justify properties of addition and subtraction.[22]
b. Produce full sets of related equations for addition and subtraction, as in the set $5 + 3 = 8$, $3 + 5 = 8$, $8 = 5 + 3$, $8 = 3 + 5$, $8 – 5 = 3$, $8 – 3 = 5$, $3 = 8 – 5$, $5 = 8 – 3$.
c. Solve up to two-step addition/subtraction word problems with whole numbers and whole number quantities within 100.[23]

Base Ten Computation

Core Standards · Students understand that:

1. A three-digit number is made up of hundreds, tens and ones units. Digits in each place are worth ten times as much as digits in the place to the right.
2. Comparison of numbers is decided by the leftmost digit, with subsequent digits breaking ties.
3. Three-digit numbers can be expanded into sums of hundreds, tens and ones units. In adding or subtracting, one adds or subtracts the units of each size; regrouping might be needed to write a total in standard form if there are too many of a unit, or to get enough of a unit to subtract from it.
4. The scheme for regrouping is the same at each place, because each unit is composed of ten of the smaller unit.

Core Standards · Students can and do:

a. Compare and order numbers to 1,000.

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[21] In join and separate problems, there is change over time. In part-part-whole problems, two quantities make up a whole in a static situation. Compare problems involve two quantities and the difference between them. Compare problems add specificity to the notions of greater than and less than.

[22] Include properties such as that the sum is the same when multiple addends are added in a different order; if adding two numbers gives a certain sum, then subtracting one of the addends from the sum results in the other addend; that if more is subtracted from a number, the difference is decreased and if less is subtracted the difference is increased; that in an addition problem, each addend can be taken apart and the parts can be recombined in any order without changing the sum.

[23] Include join, separate, part-part-whole, and compare problems, with unknowns in all positions. Represent these situations with equations that use a small square or a ? for the unknown.

b. Given a three-digit number, quickly find 10 more or 10 less than the number, and quickly find 100 more or 100 less than the number.
c. Rapidly add and subtract within 20.[24]
d. Add and subtract three-digit numbers to three-digit numbers using strategies based on place value, Properties of Arithmetic, or the inverse relationship between addition and subtraction.
e. Add and subtract three-digit numbers using an algorithm[25] based on place value and regrouping, such as the standard algorithm.
f. Explain why addition and subtraction strategies and algorithms work, using place value and the Properties of Arithmetic (including explanations supported by drawings or objects).

Quantity and Measurement

Core Standards · Students understand that:

1. 1 inch, 1 foot, 1 centimeter and 1 meter are conventionally defined lengths that allow standardized length measurements.
2. When measuring a length, if a smaller unit is chosen, more units must be iterated to measure the length in those units. But the length of an object itself does not depend on the choice of unit.
3. Units can be decomposed into smaller units, e.g. a foot contains 12 inches and a meter contains 100 centimeters. A small number of long units might form a greater total length than a large number of small units.
4. Sharing a circle or rectangle fairly among 2-6 shares creates equal parts, each of which is a single unit. Copying one unit by the number of pieces measures the whole in terms of the units.
5. A half, a third, or a quarter of a given rectangle encloses the same amount of space regardless of its shape.

Core Standards · Students can and do:

a. Measure, compare and estimate whole-unit lengths in units of inches, feet centimeters and meters.
b. Construct a number line with an origin (0) and a unit (1), marking off whole numbers one unit distance apart. Use a number line to represent sums and differences; determine lengths of intervals on the number line.
c. Decompose circles and rectangles into 2-6 equal parts. Describe the parts using the words “halves,” “thirds,” “half of,” “a third of,” etc. Describe the wholes as 2-6 times as large as the parts.
d. Construct a number line to 100 using tens-unit lengths, showing ones-unit lengths within a decade of interest. Explain regrouping by composing and decomposing concrete lengths.
e. ♦ Draw a bar graph (with single-unit scale) to represent a data set with several categories. Solve simple part-part-whole and compare problems using information presented in a bar graph.[26]
f. ♦ Identify correspondences in different representations of a data set with several categories.
g. Solve word problems involving dollar bills, quarters, dimes, nickels and pennies.

Shapes

Core Standards · Students understand that:

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[24] Acceptable strategies include: mental strategies such as making a ten, use of fingers to assist in rapid counting-on, and producing sums or differences from memory.
[25] Glossary: Algorithm. A step by step routine that always gives some answer, rather than ever giving no answer; that always gives the right answer, and never gives a wrong answer; that can always be completed in a finite number of steps, rather than in an infinite number of steps; and that applies to all problems of a given type (e.g., adding any two multidigit whole numbers, or bisecting any angle). Cf. Wikipedia’s “effective procedure,” from which this definition is adapted.
[26] For part-part-whole problems, only sum-unknown problems are required to meet this standard. For compare problems, only difference-unknown problems are required to meet the standard.

1. A given category of shapes (e.g., triangles) can be divided into subcategories (e.g., isosceles triangles) on the basis of special properties. Conversely, different classes of shapes (e.g., squares and rectangles) can be united into a larger category (e.g., quadrilaterals) on the basis of shared properties.

Core Standards · Students can and do:

a. Draw and identify equilateral triangles, isosceles triangles,[27] squares and rectangles.[28]
b. Recognize squares and rectangles as examples of quadrilaterals; draw examples of quadrilaterals that are neither squares nor rectangles.
c. Draw and identify radii and diameters of a circle.
d. Recognize objects that resemble spheres, cylinders and rectangular prisms.

[27] Students at this grade need not understand that equilateral triangles are isosceles.
[28] Students at this grade need not understand that squares are rectangles.

Mathematics: Third Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Operations and the Problems They Solve

Core Standards · Students understand that:

1. Multiplication and division apply to situations of equal grouping, fair sharing, measuring, and comparing ("times as much").
   - An equation of the form $a \times b = n$ applies to a situation in which a groups of b things each make n things in all, or in which a copies of a continuous quantity of size b form a continuous quantity of size n. (See table for examples.)
   - An equation of the form $n \div a = b$ tells how many things, b, are in each group when n things are divided equally into a groups, or tells how large a quantity b results when a continuous quantity of size n is shared fairly into a shares. (See table for examples.)
   - An equation of the form $n \div b = a$ tells how many groups, a, result when n things are divided into equal groups of b things each, or tells how many fair shares, a, a quantity of size n yields when each share has size b. (See table for examples.)
   - Two quantities can be compared by multiplication or division. An equation of the form $a \times b = n$ means n is a times as much as b and b times as much as a.
2. Multiplication is commutative: The total number of things in a groups of b things each is the same as the total number of things in b groups of a things each, that is, $a \times b = b \times a$. Likewise, a copies of a continuous quantity of size b are equal in size to b copies of a continuous quantity of size a.
3. The area of a rectangle with whole number side lengths can be calculated by multiplying because the rectangle can be decomposed into equal rows (or columns) of unit squares.
4. Multiplication and division are inverse operations; that is $35 \div 7$ can be found by thinking $5 \times 7 = 35$. When any two of the numbers in a multiplication or division equation are known, the unknown number can be found.

	$3 \times 6 = 18$	**$18 \div 3 = 6$**	**$18 \div 6 = 3$**
Collections	3 rows of apples with 6 apples in each row are 18 apples.	If 18 apples are arranged into 3 equal rows, each row will have 6 apples in it.	If 18 apples are arranged into equal rows of 3 apples, there will be 6 rows.
Continuous Quantities	If you have enough ribbon to make 6 bows, then 3 times as much ribbon will make 18 bows.	If you have enough ribbon to make 18 bows and share the ribbon fairly among 3 kids, then each kid has enough ribbon to make 6 bows.	If each kid wants to make 6 bows and there's enough ribbon to make 18 bows, then 3 kids can make bows.

Core Standards · Students can and do:

a. Use representations (objects, pictures, story contexts) to describe and justify properties of multiplication and division.[29]
b. Solve simple multiplication and division word problems involving equal groups, length and area.
c. Solve up to two-step word problems involving the four operations with whole numbers and whole number quantities. (Whole number quotients only)
d. Solve multiplicative comparison problems with whole numbers (problems involving the notion of "times as much").
e. ♦ Draw a scaled bar graph to represent a data set with several categories. Solve "how many more"/"how many less" problems (two-step problems) using information presented in scaled bar graphs.[30]

Base Ten Computation

Core Standards · Students understand that:

1. Patterns in the multiplication table can be explained by the Properties of Arithmetic. For example, the distributive property explains why, for any row, the entries in the 7 column are the sums of the entries in the 5 and 2 columns.

2. The Properties of Arithmetic can be used to derive new multiplications and divisions from known ones.

Core Standards · Students can and do:

a. Explain strategies for multiplying and dividing that use the Properties of Arithmetic and properties of the base ten system.
b. Rapidly multiply and divide within 100.[31]
c. Produce full sets of fact families for multiplication and division, as in the set $6 \times 7 = 42$, $7 \times 6 = 42$, $42 = 7 \times 6$, $42 = 6 \times 7$, $42 \div 7 = 6$, $42 \div 6 = 7$, $6 = 42 \div 7$, $7 = 42 \div 6$.
d. Find the factor pairs for a given number, as in the factor pairs for the number 42: {42, 1}, {21, 2}, {14, 3}, {7, 6}.

Fractions

Core Standards · Students understand that:

1. When a whole, 1, is divided into b equal parts, the size of the parts is written $^1/_b$. To show $^1/_b$ of something, divide the thing into b equal parts.

2. For a whole number a and a positive whole number b, $^a/_b$ is defined as a copies of $^1/_b$.[32] This can be thought of as the sum $^1/_b + {}^1/_b + \ldots + {}^1/_b$ (with a summands).

3. Whole numbers can be written as fractions, as in $^b/_b = 1$, $^n/_1 = n$, and cases such as $^{(4 \times 7)}/_4 = 7$.

4. Fractions are numbers and can be seen as lengths on a number line.[33]

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[29] Include properties such as that the product is the same when the order of the factors is changed; that multiplication problems involving 1-digit numbers can be solved by breaking one factor apart additively and multiplying each part by the other factor; and that multiplying a quantity by a number, then dividing by the same number, leaves the original quantity unchanged.

[30] Include single-unit scales and multiple-unit scales. For multiple-unit scales, all counts should be evenly divisible by the scale factor. No count should represent more than ten of the scale unit, and no scale unit should represent more than ten counts.
[31] A variety of mental strategies are acceptable, including derived fact strategies and producing products or quotients from memory.
[32] This includes fractions greater than 1. For example, $^{17}/_5$ is 17 copies of $^1/_5$.
[33] For example, $^{17}/_5$ is 17 copies of the subinterval $^1/_5$ laid end to end.

5. Two fractions are equal when they represent the same portion of a whole, or when they have the same length on a number line. One fraction is greater than another when it represents a greater portion of the whole than the other, or lies to the right of the other on the number line.

6. Given two unit fractions, the fraction with the larger denominator is smaller, because dividing a whole into a larger number of parts leads to smaller parts.

7. Fractions with the same denominator can be added or subtracted by adding or subtracting the units indicated by the unit fraction. For example, $2/3 + 4/3$ is 2 copies of $1/3$ plus 4 copies of $1/3$, or 6 copies of $1/3$ in all, that is $6/3$.

8. The decimal 0.1 denotes the fraction $1/10$, 0.2 denotes $2/10$, and so on through 0.9, which denotes $9/10$.

Core Standards · Students can and do:

a. Use fractions to describe quantities and parts of wholes.
b. Compare and order fractions with equal numerators or equal denominators, including in contextual situations, using the fractions themselves, bar strip drawings, number line representations, and area models.
c. Reason about fractions to establish equivalences between fractions with unlike denominators 2, 3, 4 and 6 (e.g. $1/2 = 2/4$, $4/6 = 2/3$).
d. Add and subtract fractions with like denominators.
e. Solve word problems that involve adding, subtracting, ordering and comparing fractions.
f. Represent fractions of the form $a/10$ in decimal notation; compare and order to tenths in decimal notation.

Quantity and Measurement

Core Standards · Students understand that:

1. A unit of measure can be partitioned into equal-sized parts, whose sizes can be represented as fractions of the unit.

2. The area of a closed plane figure is a measure of how much space it encloses. A square with side length 1 unit is said to enclose “one square unit” of area.

3. The area of a closed plane figure can be measured (expressed numerically) by the number of square units that fit inside it with no gaps or overlaps.

4. Area is a model for multiplication because tiling a rectangle with unit squares shows that a rectangle a units long by b units wide encloses an area of $a \times b$ square units.

Core Standards · Students can and do:

a. Measure lengths using rulers marked with halves and fourths of inches. Make a dot plot to show repeated measurements.
b. Convert compound units to a smaller or a larger unit, and solve problems involving mixed units (feet and inches, yards and feet).
c. Using customary units, demonstrate and justify correct processes for measuring, comparing, and estimating length, mass, capacity, and durations of time, including unit selection, partitioning and iterating units, and transitivity.
d. Compute perimeters of polygons by adding given side lengths, and find an unknown length in a polygon given the perimeter and all other side lengths. Represent these problems with equations involving a symbol for the unknown quantity.
e. Determine and compare areas by counting square units (improvised units, cm^2, m^2, in^2, ft^2).
f. Compute elapsed time and solve problems involving elapsed time (to the nearest minute).

Mathematics: Fourth Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Operations and the Problems They Solve

Core Standards · Students understand that:

1. Quantities in a problem might be described with whole numbers, fractions or decimals; the operations used to solve the problem depend on the relationships between the quantities whatever numbers are involved.
2. The distributive property (of multiplication over addition) relates addition and multiplication. The distributive property can be shown numerically and visually, using arrays and area models.

Core Standards · Students can and do:

a. Solve multistep word problems involving the four operations with whole numbers.[34]
b. Estimate answers to computations and compute mentally to assess reasonableness of results.
c. Solve problems that involve comparing, ordering, adding and subtracting fractions with like denominators. Compare fractions to benchmark fractions.
d. Solve problems that involve comparing and ordering decimal numbers to hundredths. Compare decimals to benchmark decimals.
e. ♦Make a table from given data, ask and answer questions about data in a table, solve multi-step problems using information presented in tables, and find patterns in tables.[35]

Fractions

Core Standards · Students understand that:

1. The fraction a/b can be written as $a \times 1/b$ because a/b is a copies of $1/b$.
2. When a identical things are divided into b equal parts, each of a things contributes $1/b$. So $a \div b = a/b$.[36]
3. A fraction can be multiplied by a whole number as $n \times a/b = n{\times}a/b$. For example, $3 \times 2/5$ can be seen as 3 groups of 2 unit fractions $1/5$.[37]
4. A decimal of two digits stands for a sum of fractions whose denominators are 10 and 100. For example, 0.34 stands for $3/10 + 4/100$.

Core Standards · Students can and do:

a. Reason about fractions to establish equivalences between related fractions[38] (e.g. $3/10 = 30/100$, $9/12 = 3/4$).

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34 Use the properties of multiplication (commutative, associative, identity) or the inverse relationship between multiplication and division (multiplying a number by b then dividing by b, and vice versa, leaves the number unchanged) to make sense of single digit multiplication and division situations and solve problems.
35 Include tables with data from proportional relationships.
36 This definition agrees with previous understandings of division in cases like 28 ÷ 7 (i.e., when a is a multiple of b), but also gives meaning to quotients such as 3 ÷ 4 or 7 ÷ 2.
37 Using the Properties of Arithmetic, $n \times a/b = n \times (a \times 1/b) = (n \times a) \times 1/b = (n{\times}a)/b$.

b. Add and subtract related fractions in simple cases within one whole (e.g. $1/2 + 1/4$, $3/10 + 4/100$).
c. Solve word problems posed with whole numbers that have fractional answers.
d. Represent multiplication of whole numbers by fractions and fractions by whole numbers, using length and area models.
e. Solve word problems involving multiplying fractions by whole numbers and multiplying whole numbers by fractions.[39]
f. Use decimals to hundredths to describe quantities and parts of wholes, compare and order decimals to hundredths, and write fractions of the form $a/10$ or $a/100$ in decimal notation.
g. Round decimals (to hundredths) to the nearest whole number.
h. Solve addition and subtraction story problems involving fractions with related denominators (situations familiar from whole number work).

Base Ten Computation

Core Standards · Students understand that:

1. The product of a one-digit number times a multidigit number is the sum of the products of the one-digit number times each place value component. This is an instance of the distributive property.
2. Multi-digit multiplication algorithms can be derived and explained by decomposing numbers into their place value components and applying the distributive property.
3. Digits in each place are worth ten times as much as digits in the place to the right and a tenth as much as digits to the left; comparison of numbers is decided by the leftmost digit, with subsequent digits breaking ties.
4. Given whole numbers a and b, find whole numbers Q and R so that $a = Q \times b + R$. For example, given 325 and 7, express 325 in the form $325 = 46 \times 7 + 3$.

Core Standards · Students can and do:

a. Demonstrate place value understanding for whole numbers to 1,000,000 and compare numbers within this range.
b. Round whole numbers to the nearest 10 or 100 and use rounding to estimate computations.
c. Multiply single place numbers (to 9000) by single digit numbers.[40]
d. Multiply two-, three- and four-digit numbers by single-digit whole numbers, and multiply two-digit numbers by two-digit numbers, using strategies based on place value, Properties of Arithmetic, or the inverse relationship between multiplication and division.
e. Multiply two-digit numbers by two-digit numbers using an algorithm based on place value and regrouping, such as the standard algorithm.
f. Divide two-, three- and four-digit numbers by single-digit numbers, with or without remainder. In the case of remainders, express results in the form of an equation, as in $325 = 46 \times 7 + 3$.
g. Explain why multiplication and division strategies and algorithms work, using place value and the Properties of Arithmetic (including explanations supported by drawings or objects).

Quantity and Measurement

Core Standards · Students understand that:

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[38] Glossary: Related fractions. Two fractions are *related* if one denominator is a factor of the other. (See Ginsburg, Leinwand and Decker (2009), *Informing Grades 1-6 Mathematics Standards Development: What Can Be Learned from High-Performing Hong Kong, Korea, and Singapore?*, Table A1, p. A-5, grades 3 and 4.)
[39] Include sharing multiple continuous wholes *a* fairly among *b* people, naming an individual share as a/b. For example 5 meters of pink ribbon shared among 3 people results in $5/3$ meters each.
[40] Glossary: Single-place number. The numbers that result when a whole number between 1 and 9 (inclusive) is multiplied by the numbers 10, 100, 1000, etc.

1. Area is additive: If a figure is decomposed into several pieces, then the area of the whole figure can be found by adding the areas of the pieces (expressed in common units).
2. An angle is two rays with a common endpoint, and is measured by the relative amount of a circle that you trace when turning from one ray to the other.
3. A one-degree angle turns through 1/360 of a circle, where the circle is centered at the origin of the rays; the measure of an angle is the number of one-degree angle turned with no gaps or overlaps.

Core Standards · Students can and do:

a. Apply the formula for area of squares and rectangles. Measure and compute whole-square-unit areas of objects and geometric figures decomposable into rectangles.[41]
b. ♦ Make a dot plot to show repeated measurements in common fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in dot plots (e.g., finding the difference in length between the longest and shortest specimens in an insect collection).
c. Draw scales (number line representations) of problem situations involving length, height and distance including fractional units or decimal numbers.
d. Find one dimension of a rectangle given the other dimension and its area or perimeter; find the length of one side of a square given its area or perimeter. Represent these problems with equations involving a symbol for the unknown quantity.
e. Measure angles in whole-number degrees using a protractor; sketch angles of specified measure. Find the measure of a missing part of an angle, given the measure of the angle and the measure of a part of it; represent these problems with equations involving a symbol for the unknown quantity.

Shapes

Core Standards · Students understand that:

1. Shapes can be analyzed and classified using concepts of parallelism, perpendicularity and angle measure.

Core Standards · Students can and do:

a. Draw points, lines, line segments, rays and angles; identify these in geometric figures.
b. Associate angles of a quarter turn (subtending ¼ of a circle) with angle measure 90º, a half turn (½ of a circle) with angle measure 180º, ¾ turn (¾ of a circle) with angle measure 270º, and a full turn (complete circle) with angle measure 360º.[42]
c. Draw perpendicular and parallel lines; identify these in geometric figures.
d. Identify right angles and angles smaller than/greater than a right angle in geometric figures; recognize right triangles.
e. Given a quadrilateral, say whether it is a square, whether it is a rectangle, and whether it is a parallelogram (with an understanding that a given shape may fit more than one category).

[41] using one-digit or two-digit numbers times two-digit numbers
[42] From Singapore Primary 4

Mathematics: Fifth Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Fractions

Core Standards · Students understand that:

1. Fractions a/b and $(n\times a)/(n\times b)$ are equal: for $1/b$ is n copies of $1/(n\times b)$, so a/b is $n \times a$ copies of $1/(n\times b)$. Example: $1/3$ is 4 copies of $1/12$, so $2/3$ is 8 copies of $1/12$; thus $2/3 = 8/12$.
2. Fractions can be added or subtracted by replacing each with an equal fraction so that the resulting fractions have the same denominator. Example: $2/3 + 5/4 = 8/12 + 15/12 = 23/12$.
3. Multiplying unit fractions gives a new unit fraction with denominator equal to the product of the initial denominators. For example, $1/3 \times 1/2 = 1/(3\times 2)$. The product $1/3 \times 1/2$ is 1 part when a whole of size $1/2$ is divided into 3 parts, i.e. it is "$1/3$ of $1/2$."[43]
4. Multiplying unit fractions can be extended to multiplying fractions in general. For example, $2/3 \times 4/5$ can be seen as 2 groups of 4 unit fractions $1/15$, hence the product is $8/15$.[44] The product $2/3 \times 4/5$ is 2 parts when a whole of size $4/5$ is divided into 3 parts, i.e. it is "$2/3$ of $4/5$."[45]
5. Dividing a unit fraction $1/b$ by a whole number n gives a unit fraction with denominator $n \times b$, because when $1/b$ is divided into n equal parts, the size of each part is $1/(n\times b)$. For example, $1/3 \div 2 = 1/6$.
6. Dividing a whole number n by a unit fraction $1/b$ gives a whole number $n \times b$, because, as there are b units of $1/b$ in 1, there are $n \times b$ units of $1/b$ in n. For example, $2 \div 1/3 = 6$.
7. A mixed number stands for the sum of its whole number part and a fractional part less than 1. A mixed number can be written as a fraction greater than 1, such as $17/5$. This equivalence can be shown using area, length, and number line models.
8. The ratio of two whole number quantities a and b, written $a{:}b$ or a/b, is a multiplicative comparison telling how much of one quantity there is for a given amount of the other, or how many times as much one is than the other.[46]

Core Standards · Students can and do:

a. Use area models and length models (such as strip drawings or the number line) to represent multiplication of fractions, division of unit fractions by whole numbers, and division of whole numbers by unit fractions.
b. Multiply fractions, divide unit fractions by whole numbers, and divide whole numbers by unit fractions, and solve word problems involving these operations.
c. Divide whole numbers by single digit decimals by seeing that they are fractions with denominator 10 or 100.
d. Rename fractions and mixed numbers to equivalent forms and identify equivalent fractions.
e. Compare and order fractions and mixed numbers with like or unlike denominators, including in contextual situations, using the fractions themselves, strip drawings or number line representations, and area models. Describe the size of fractional quantities with reference to the problem situation.
f. Make tables of equal ratios relating whole number quantities, and find missing values in the tables. Plot pairs of values on the coordinate plane. Example

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[43] On the number line, $1/n \times 1/d$ is 1 part when the interval from 0 to $1/d$ is divided into n parts. This is the same as 1 part when the interval from 0 to 1 is divided into $n \times d$ parts, and thus $1/n \times 1/d = 1/n\times d$.
[44] Using the Properties of Arithmetic, $2/3 \times 4/5 = (2 \times 1/3) \times (4 \times 1/5) = (2 \times 4) \times (1/3 \times 1/5) = (2 \times 4) \times 1/3\times 5 = 2\times 4/3\times 5$.
[45] On a number line, $m/n \times c/d$ means m parts when the interval from 0 to c/d is divided into n parts.
[46] For example, in a mixture of 5 cups of flour and 2 cups of sugar, the ratio is 5 cups flour to 2 cups sugar. There is $5/2$ times as much flour as sugar (equivalently, 2 $1/2$ times as much or 2.5 times as much).

Cups of Flour	5	10	?	20
Cups of Sugar	2	?	6	8
Flour:Sugar (fraction form)	$5/2$	$10/4$	?	$20/8$
Flour:Sugar (decimal form)	?	?	?	?

Hours of Snowfall	5	10	?	20
Inches of Snow	2	?	6	8
Inches:Hours (fraction form)	$2/5$	$4/10$	?	$8/20$
Inches:Hours (decimal form)	?	?	?	?

Base Ten Computation

Core Standards · Students understand that:

1. The standard division algorithm is based on successively finding the largest single digit multiple of the divisor that is less than the dividend, regrouping to the next lower unit if necessary, and then subtracting the multiple and repeating to find the next digit in the quotient.
2. The division algorithm can be used to express a fraction in decimal form by carrying the division into the decimal places.
3. The features of the place value system for whole numbers extend to the decimal positions and the combined system is symmetric around the ones place.
4. In adding or subtracting decimal numbers, one operates separately with the units of each size, except when regrouping is needed; the scheme for regrouping is the same at each place, because each unit is composed of ten of the next smaller unit.
5. Numbers in decimal notation can be shown on the number line by dividing and sub-dividing the unit intervals as many times as needed to locate the number. This process can be visualized as zooming in on the number line.

Core Standards · Students can and do:

a. Divide two, three and four digit numbers by two digit numbers, with remainder, using an algorithm based on place value and regrouping, such as the standard algorithm. In the case of remainders, express results in the form of an equation, as in $145 = 11 \times 13 + 2$.
b. Understand very large and very small numbers (from millionths to hundreds of millions); round very large numbers.
c. Quickly find 0.1 more than a number and less than a number; 0.01 more than a number and less than a number; and 0.001 more than a number and less than a number.
d. Add and subtract decimals using an algorithm based on place value and regrouping, such as the standard algorithm, and solve problems involving these operations.
e. Write fractions in decimal notation for denominators 2, 3, 4, 5, 6, 8, 10 and 100.
f. Explain why strategies and algorithms for decimals work, using place value and the Properties of Arithmetic (including explanations supported by drawings or objects).

Quantity and Measurement

Core Standards · Students understand that:

1. The volume of a solid figure is a measure of how much space it encloses. A cube with side length 1 unit is said to contain "one cubic unit" of volume. The volume of a solid figure can be measured (expressed numerically) by the number of cubic units that fit inside it with no gaps or overlaps.
2. Packing a rectangular prism with unit cubes and decomposing the prism into layers shows that a rectangular

prism ℓ units long by w units wide by h units tall contains a volume $V = \ell \times w \times h$ cubic units. The base of the prism has area $A = \ell \times w$ square units, and the prism can be viewed as h layers, each containing $\ell \times w$ cubic units,

so the volume of the prism can also be expressed as $V = A \times h$ cubic units.

3. Volume is additive: If a solid figure is decomposed into several pieces, then the volume of the whole figure can be found by adding the volumes of the pieces (expressed in common units).
4. Quantities with like units can be added or subtracted giving a sum or difference with the same unit; quantities with unlike units can be multiplied or divided giving products or quotients with derived units.
5. The ratio of a length, area or amount to another length, area or amount is the same regardless of the size of the unit used for measurement.
6. The number line is a scale that can be used to show units such as pounds, liters, etc.

Core Standards · Students can and do:

a. Measure and compute whole-cubic-unit volumes for rectangular prisms and for objects well described as rectangular prisms.
b. Convert among different-sized standard measurement units within a given measurement system (e.g. feet to yards, centimeters to meters) and use conversion to solve story problems.
c. Form ratios of lengths, areas, and other quantities, including when quantities being compared are measured in different units.
d. Solve word problems involving addition, subtraction, multiplication and/or division using quantities expressed as whole numbers, fractions, or decimals with measurement units.
e. Solve multi-step problems involving units of weight, capacity, money, volume and area.

Coordinate Geometry

Core Standards · Students understand that:

1. A pair of perpendicular number lines (“axes”) defines a coordinate system. A given point in the plane has a separate position along each of the two axes; the two positions of the point are called its coordinates.
2. Graphs on coordinate axes can be used to make sense of relationships among quantities in complex problems.

Core Standards · Students can and do:

a. Graph points in the first quadrant the coordinate plane, and read off the coordinates of graphed points.[47]
b. Determine the lengths of horizontal and vertical segments in the plane, given the coordinates of their endpoints.
c. ♦ Collect data on continuous covarying quantities and display the data in a line graph with broken lines; distinguish bar graphs from line graphs; ask and answer questions from line graphs, including comparisons of ratios.

Statistics

Core Standards · Students understand that:

[47] The axes should sometimes represent dimensioned quantities, and the units of measure should not always be the same for both axes. Coordinates may be whole numbers, fractions or decimals.

1. Data are collected purposefully to answer a predefined question (e.g., "How tall are the fifth graders in our school?")
2. A set of data typically shows variability—not all of the values are the same—and yet the values also typically show some tendency to cluster. Identifying a "center" for a data set is a way to describe its many values using a single number.
3. The median is a measure of center in the sense that approximately half the data values are less than median, while approximately half are greater.
4. Variation in a data set can be measured by the range and by typical deviations from the center.

Core Standards · Students can and do:

a. Collect data to answer a predefined question about a measurement quantity. Make a dot plot to display the data, and describe the data using the median and typical deviations from the it.

Mathematics: Sixth Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Ratios and Proportional Relationships

Core Standards · Students understand that:

1. Multiplicative comparisons can be extended from whole numbers to fractions and decimals. When the ratio q/m is formed, or when q is r times as much as m, the numbers q, r and m can be fractions or decimals.
2. p% of a quantity means $p/100$ times as much as the quantity. The number p can be a fraction or decimal, as in 3.75%.
3. A unit rate is the multiplicative factor relating the two quantities in a ratio. Two quantities q and m can be compared by $q = r \times m$, where the unit rate r tells how much q per m.
4. Given two quantities in a ratio (e.g. distance and time), finding the unit rate produces a new type of quantity (e.g. speed).

Core Standards · Students can and do:

a. Solve for an unknown quantity in a problem involving two equal ratios.
b. Find a percentage of a quantity; solve problems involving finding the whole given a part and the percentage.
c. Solve unit rate problems including unit pricing and constant speed. (See table.)

$D = s \times T$	$D \div T = s$	$D \div s = T$
A car driving at a speed of 30 miles per hour for 6 hours travels a distance of 180 miles.	If a car drives 180 miles for 6 hours at a constant speed, that speed is 30 miles per hour.	When a car drives 180 miles at a speed of 30 miles per hour, the trip takes 6 hours.

d. Represent unit rate problems on a coordinate plane where each axis represents one of the two quantities involved, and find unit rates from a graph. Explain what a point (x, y) means in terms of the situation, with special attention to the points (0, 0) and $(1, r)$ where r is the unit rate.

The Number System

Core Standards · Students understand that:

1. The Properties of Arithmetic govern operations on all numbers.
2. Division of fractions follows the "invert and multiply" rule because multiplication and division are inverse operations. For example, $(2/3) \div (5/7) = 14/15$ because $(14/15) \times (5/7) = 2/3$.
3. Every nonzero fraction has a unique multiplicative inverse,[48] namely its reciprocal. Division can be defined as "multiplying by the multiplicative inverse." Then $(2/3) \div (5/7) = 14/15$ because the division symbol indicates multiplication by the multiplicative inverse.
4. A two-sided number line can be created by reflecting the fractions across zero. Numbers located to the left of zero on the number line are called negative numbers and are labeled with a negative sign.

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[48] Glossary: Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: $3/4$ and $4/3$ are multiplicative inverses of one another because $3/4 \times 4/3 = 4/3 \times 3/4 = 1$.

5. Two different numbers, such as 7 and –7, that are equidistant from zero are said to be opposites of one another. The opposite of 7 is –7 and the opposite of –7 is 7. The opposite of the opposite of a number is the number itself. The opposite of 0 is 0. The operation of attaching a negative sign to a number can be interpreted as reflecting the number across zero on the number line.

6. The absolute value of a number is its distance from zero on the number line. For any positive number q, there are two numbers whose absolute value is q, namely q and $-q$.

7. The absolute value of a signed quantity (e.g. account balance, elevation) tells the size of the quantity irrespective of its sense (debit or credit; above or below sea level).

8. Comparison of numbers can be extended to the full number system. The statement $p > q$ means that p is located to the right of q on the number line, while $p < q$ means that p is located to the left of q on the number line. The statement $p > q$ does not mean $|p| > |q|$.

Core Standards · Students can and do:

a. Divide fractions, and divide finite decimals by expressing them as fractions.
b. Solve problems requiring arithmetic with fractions presented in various forms, converting between forms as appropriate and estimating to check reasonableness of answers.
c. Find and position rational numbers[49] on the number line.
d. Use rational numbers to describe quantities such as elevation, temperature,[50] account balance and so on. Compare these quantities using > and < symbols and also in terms of absolute value.
e. Graph points and identify coordinates of points on the Cartesian coordinate plane in all four quadrants.

Statistics

Core Standards · Students understand that:

1. The mean is a measure of center in the sense that it is the balance point; the mean is the value each data point would take on if the total value of all the data points were redistributed fairly.

2. When the mean and median of a data set differ substantially, both measures should be provided, and the difference explained in terms of the data values.

Core Standards · Students can and do:

a. Collect data to answer a predefined question about a measurement quantity. Make a dot plot to display the data, and describe the data using measures of center and measures of variation.[51]

Geometry

Core Standards · Students understand that:

1. Triangles and parallelograms can be dissected and reassembled into rectangles with the same area; this leads to a formula for area in terms of base and height.

2. Polygons can be dissected into triangles in order to find their area.

Core Standards · Students can and do:

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[49] Glossary: Rational number. A number expressible in the form $^a/_b$ for integers a and $b \neq 0$. The rational numbers include positive and negative integers, positive and negative fractions, and 0.

[50] A caution for temperature problems: The rational numbers are not a good model for a temperature scale. There is no temperature that solves the equation $T + 1000^{\circ}\mathrm{C} = 0$.

[51] Data sets should include fractional values at this grade but not negative values.

a. Find the area of right triangles, other triangles, special quadrilaterals, and polygons (by dissection into triangles and other shapes).
b. Find surface area of cubes, prisms and pyramids (include the use of nets to represent these figures).
c. Solve problems involving area, volume and surface area of objects.
d. Examine the relationship between volume and surface area. Exhibit rectangular prisms with the same surface area and different volume, and with the same volume and different surface area.
e. Use exponents and symbols for square roots and cube roots to express the area of a square and volume of a cube in terms of the side length, and to express the side length in terms of the area or volume.

Expressions and Equations

Core Standards · Students understand that:

1. A number that is the result of a sequence of operations with other numbers can be expressed in different ways using conventions about order of operations and parentheses, rules for working with fractions, and the Properties of Arithmetic. All such expressions are equivalent.
2. A letter is used to stand for a number in an expression in cases where one doesn't know what the number is, or where, for the purpose at hand, it can be any number in the domain of interest. Such a letter is called a variable.
3. An equation is a statement that two expressions are equal, and a solution to an equation is a value of the variable (or a set of values for each variable if there is more than one variable) that makes the equation true.

Core Standards · Students can and do:

a. Represent an unknown number using a letter in simple expressions such as $y + 2$, $y - 3$, $6 + y$, $5 - y$, $3y$, $y/2$, and $(3 \pm y)/5$.
b. Interpret $3y$ as $y + y + y$ or $3 \times y$, $y/2$ as $y \div 2$ or $1/2 \times y$, $(3 \pm y)/5$ as $(3 \pm y) \div 5$ or $1/5 \times (3 \pm y)$.[52]
c. Evaluate simple expressions when values for the variables in them are specified (exclude expressions with a variable in denominator).
d. Choose variables to represent quantities in a word problem and construct simple equations to solve the problem by reasoning about the quantities.
e. Solve equations of the form $x + p = q$ (for $p < q$) and $px = q$ where p and q are fractions.

[52] From Singapore Secondary 1

Mathematics: Seventh Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Ratios and Proportional Relationships

Core Standards · Students understand that:

1. Two variable quantities x and y are said to be proportional to one another if the ratio x/y is always equal to the same quantity k, so that $y = kx$. The constant k is the unit rate, and tells how much of y per unit of x.

Core Standards · Students can and do:

a. Compare proportional relationships represented in different ways (e.g. compare a graph to an equation to determine which of two objects has greater speed).
b. Decide whether two quantities that vary together have a proportional relationship, analyze proportional relationships using the unit rates that characterize them, and solve word problems involving proportional relationships.
c. Plot pairs (x, y) from a proportional relationship $y = kx$, and pass a straight line through them and the origin. Observe that increases in y are proportional to increases in x, and calculate [increase in y]/[increase in x] $= k$.

The Number System

Core Standards · Students understand that:

1. On the number line, the sum $p + q$ is defined to be the number lying distance $|q|$ from p, to the right of p if q is positive and to the left of p if q is negative. A number and its opposite are additive inverses (add to zero).[53]
2. Sums of signed numbers can be computed using the Properties of Arithmetic.[54]
3. The additive inverse of a sum is the sum of the additive inverses.[55]
4. Subtraction is defined as adding the additive inverse. This definition of subtraction allows subtraction of rational numbers and agrees with previous understandings of subtraction with positive numbers.[56] On the number line, the difference $p - q$ lies distance $|q|$ from p, to the left of p if q is positive and to the right of p if q is negative.
5. The absolute value of $p - q$ equals the distance between p and q on the number line.
6. Products of signed numbers can be computed using the Properties of Arithmetic.[57] In particular, multiplying a number by -1 produces its additive inverse.[58]

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[53] Glossary: Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: $3/4$ and $-3/4$ are additive inverses of one another because $3/4 + (-3/4) = (-3/4) + 3/4 = 0$.
[54] For example, $7 + (-3) = 4$ because $7 + (-3) = (4 + 3) + (-3) = 4 + [3 + (-3)] = 4 + [0] = 4$. And $(-2) + (-3) = -5$ because $5 + [(-2) + (-3)] = (2 + 3) + [(-2) + (-3)] = [2 + (-2)] + [3 + (-3)] = [0] + [0] = 0$ so $(-2) + (-3)$ is the additive inverse of 5, that is -5.
[55] For example, $-(6 + -2) = (-6) + 2$ because $[6 + (-2)] + [(-6) + 2] = [6 + (-6)] + [(-2) + 2] = [0] + [0] = 0$.
[56] For example, the subtraction $7 - 3$ means 7 plus the additive inverse of 3, i.e. $7 + (-3)$, which equals 4. The subtraction $9 - (-4)$ means 9 plus the additive inverse of -4, i.e. $9 + 4$, which equals 13.
[57] For example, $(-1) \times (-1) = 1$ because $(-1) + (-1) \times (-1) = 1 \times (-1) + (-1) \times (-1) = [1 + (-1)] \times (-1) = 0 \times (-1) = 0$.
[58] Because $(-1) \times a + a = (-1) \times a + (1) \times a = [(-1) + (1)] \times a = 0 \times a = 0$.

7. Every nonzero rational number has a multiplicative inverse. Division of rational numbers is defined as multiplying by the multiplicative inverse.
8. The operation of adding the rational number q to points on the number line is a translation; it shifts points to the right if $q > 0$, to the left if $q < 0$, and not at all if $q = 0$. The operation of adding $-q$ undoes the operation of adding q.
9. The operation of multiplying points on the number line by a positive rational number k is a dilation; it scales points further away from zero if $k > 1$, closer to zero if $k < 1$, and not at all if $k = 1$. The operation of multiplying by $1/k$ undoes the operation of multiplying by k.

Core Standards · Students can and do:

a. Explore and explain with number lines the rules for adding rational numbers, e.g., $r + s = s + r$; $r + (-s) = r - s$; $p - (q + r) = p - q - r$.
b. Use the rules of arithmetic to explore and explain with specific numbers the rules for multiplying rational numbers, e.g., 4×-5 is -5 added to itself 4 times, so equal to -20; $-3 \times (-2 + 2) = -3 \times 0 = 0$, so $-3 \times -2 = -(-3)\times 2 = -(-6) = 6$.
c. Add and subtract rational numbers, and use these operations to solve word problems (including signed quantities such as elevation, temperature, account balance, and so on).
d. Multiply and divide rational numbers, and use these operations to solve word problems (including signed quantities).

Expressions and Equations

Core Standards · Students understand that:

1. Expressing a quantity in different forms serves a purpose in analyzing quantitative situations.
2. The distributive property can be used in two directions, both to expand linear expressions, and to factor a sum of terms with a common factor.

Core Standards · Students can and do:

a. Construct algebraic expressions for simple real-world situations and generate equivalent expressions to interpret their meaning (e.g., $P + 0.05P = 1.05P$ means that "increase by 5%" is the same as "multiply by 1.05").
b. Generate equivalent expressions from a given expression, including putting linear expressions in standard form and taking out a common factor. Include expressions involving negative numbers and exponents 2 and 3.
c. Solve multi-step word problems that lead to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are rational numbers, by undoing the operations involved in producing the expression on the left, using additive and multiplicative inverses.
d. Solve simple absolute value equations of the form $|x + h| = j$ and $|x - h| = j$, where h and j are integers.
e. Read the structure in a numerical expression at a level necessary to enter it into a calculator or spreadsheet, making use of parentheses and the conventions on order of operations.

Statistics

Core Standards · Students understand that:

1. In addition to measurement variability, another source of variation in data is randomness.

Core Standards · Students can and do:

a. Collect experimental or simulation data from repeated random trials. Make a histogram showing absolute frequencies and a bar graph of relative frequencies. Discuss the patterns and make predictions for further experiments or simulations.

Probability

Core Standards · Students understand that:

1. Chance events fall along a spectrum: nearly impossible | unlikely | neither likely nor unlikely | likely | nearly certain.
2. Probability is a quantitative measure of likelihood. Probabilities are numbers lying between 0 and 1, with 0 representing impossible and 1 representing certain (in the case of a finite sample space).
3. The experimental probability of a specified outcome is the observed fraction of the outcome in a data set collected from a process involving randomness or chance.
4. In a random process, the individual outcomes are unpredictable, but patterns may emerge after repeated trials. Experimental probabilities in random experiments tend to approach stable values as more and more data is generated.
5. In a theoretical probability model, the set of distinct possible outcomes for a random experiment is called the sample space. An event is a set of sample points; a sample point may belong to several events. A specified event occurs in some fraction of the sample space. This fraction is called the theoretical probability of the event.
6. When computing theoretical probabilities, all members of the sample space are assumed equally probable. Theoretical probabilities will not match long-run experimental probabilities if this assumption is inappropriate (e.g., as in the case of a loaded die).

Core Standards · Students can and do:

a. Compute experimental probabilities from data sets, including data sets generated by simulations or sampling experiments.
b. Compute experimental probabilities to estimate theoretical probabilities when no theoretical probability model is apparent.
c. Represent sample spaces for one-stage random experiments; identify members of the sample space in which specified events occur.
d. Use a theoretical probability model to compute theoretical probabilities for one-stage random experiments, expressing theoretical probabilities as fractions, decimals and percents.
e. Compare experimental probabilities to theoretical probabilities for one-stage random experiments, examining and if feasible revising the assumptions of the theoretical model when the two conflict.

Geometry

Core Standards · Students understand that:

1. Two polygons are congruent[59] if and only if there is a correspondence between vertices so that the corresponding sides are equal and the corresponding angles are equal.
2. A plane or solid figure is similar to another if the second can be obtained from the first by a similarity transformation.[60] All ratios of lengths in the second figure to corresponding lengths in the first figure are equal to the scale factor of the dilation.

[59] Glossary: Congruent. Two plane or solid figures are *congruent* if one can be obtained from the other by a sequence of rigid motions (rotations, reflections, and translations).
[60] Glossary: Similarity transformation. A rigid motion followed by a dilation. Glossary: Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

3. Congruent figures have the same area or volume. A similarity transformation with a scale factor of k leaves angle measures unchanged, changes lengths by a factor of k, changes areas by a factor of k^2, and changes volumes by a factor of k^3.

4. Given a line in the coordinate plane not parallel to either axis, any two right triangles with legs parallel to the axes and hypotenuse on the given line are similar, and so the slope of the line (rise over run) is the same regardless of which two distinct points are used to compute it.

Core Standards· Students can & do:

a. Solve problems involving similar triangles and scale drawings (including computing actual lengths, areas and volumes from a scale drawing and reproducing a scale drawing at a different scale).
b. Explore using hands-on activities the area of non-rectangular figures and the perimeter of curvilinear figures, and the fact that a dilation of the plane changes areas by the square of the scale factor.[61]
c. Use scale factors to find lengths and areas of similar figures, including an informal derivation of the formulas relating the area, radius and circumference of a circle.
d. Give an explanation of why the volume of a cylinder is the area of the base times the height, using informal arguments involving slices.
e. Use coordinate grids to transform figures and to predict the effect of dilations, translations, rotations and reflections.
f. Use two-dimensional representations of three-dimensional objects (schematics, assembly instructions, perspective drawings and multiple views) to solve problems.
g. Explore three-dimensional figures formed by translations and rotations of plane figures through space.
h. Sketch and describe cross-sections of cones, cylinders, pyramids and prisms.

[61] Include using grids of squares with fractional side lengths to estimate area, and measuring the length of strings wrapped around the perimter.

Mathematics: Eighth Grade

Developing Coherent Understanding

[Temporarily removed for editing.]

Functions and the Situations They Model

Core Standards · Students understand that:

1. A function is a rule, often defined by an expression, that states a relationship between the values of two variable quantities.
2. A linear function models a situation where the change in one quantity is proportional to the corresponding change in the other quantity. The constant of proportionality, m, is the rate of change of the function. If x is the input and y is the output then the function is defined by $y = mx + b$ for some constant b, which is called the initial value of the function (the value of the function when x is 0).
3. The graph of a linear function $y = mx + b$ is a straight line, and the slope of the line is the function's rate of change.
4. The problem of finding where two linear functions have the same output value for a common input value leads to an equation in one variable; the solution or solutions (if any) can be visualized as the input value(s) where the graphs of the functions intersect.
5. A linear equation in one variable can be solved by successively transforming it into simpler equations with the same solutions using the Properties of Arithmetic and the Properties of Equality, until an equation of the form $x = a$, $a = a$, or $a = b$ results (where a and b are different numbers).

Core Standards · Students can and do:

a. Compare features of two or more functions that may be presented in different representations (as formulas, graphs, tables of values, or verbally).
b. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship; from two (x, y) values (including reading these from a table); or from a graph.
c. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
d. Solve linear equations with rational number coefficients, including equations that require expanding expressions using the distributive law and collecting like terms.

The Number System

Core Standards · Students understand that:

1. The number line has numbers that are not rational, such as 2π or $2 + \sqrt{3}$, called irrational numbers.
2. An irrational number can be approximated to arbitrary precision by rational numbers.
3. If $n > 0$ is an integer and $\sqrt{n}$ is not an integer, then $\sqrt{n}$ is irrational. If q is rational and r is irrational, then $q + r$ is irrational, and so is qr provided $q \neq 0$.

Core Standards · Students can and do:

a. Use rational approximations to compare the size of irrational numbers, locate them approximately on a number line and estimate the value of expressions (e.g. π^2).

Geometry

Core Standards · Students understand that:

1. Angle measures formed by a configuration of lines in a plane can often be deduced from other angle measures (e.g., vertically opposite angles, angles produced when a transversal line cuts two parallel lines).
2. The side lengths of a right triangle are related by the Pythagorean theorem.

Core Standards · Students can and do:

a. Explore and explain by hands-on activities facts about the angle sum of triangles, exterior angles, and alternate interior angles of parallel lines. Use these facts to determine the angle sum of interior angles of convex polygons, and the angle sum of exterior angles of convex polygons.[62]
b. Explore and explain using hands-on activities: parallel lines in space, line perpendicular to a line through a given point, lines perpendicular to a plane, lines parallel to a plane, the plane passing through three given points, and the plane perpendicular to a given line at a given point.
c. Use facts about angles to write and solve simple equations for an unknown angle in a figure.
d. Explain a proof of the Pythagorean theorem.[63]
e. Use the Pythagorean theorem to determine missing side lengths in right triangles and to solve problems in two and three dimensions.
f. Use the Pythagorean theorem to find the distance between two points in a coordinate system.
g. Draw (freehand, with ruler and protractor, and with technology) geometric shapes from given conditions. (Focus on constructing triangles from three measures of angles or sides, noticing when the triangle is uniquely defined, ambiguous, or impossible.)
h. Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc): copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.
i. Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Statistics

Core Standards · Students understand that:

1. Scatterplots for bivariate continuous data may reveal patterns of association between two quantities. This kind of relationship between quantities is not a functional relationship—and yet, a function might be a valuable way to describe a statistical relationship.

Core Standards · Students can and do:

a. Construct and interpret scatterplots for bivariate measurement data.
b. Describe patterns that appear in scatterplots, such as clustering, outliers, positive/negative association, linear association, nonlinear association.
c. For scatterplots that suggest a linear association, model the relationship with a linear function using an informal fitting procedure. Use the model function to solve problems in the context of the data,

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62 Use physical models, transparencies, or dynamic geometry software to make rigid motions and give informal arguments, for example, arrange three copies of the same triangle so that the three angles form a line, and give an argument in terms of transversals why this is so.
63 For example, by the method of right triangles in a square.

interpreting the slope/rate of change and intercept/initial value. Informally assess the goodness of the model by judging the closeness of the data points to the graph of the function.

Probability

Core Standards · Students understand that:

1. The framework for theoretical probability models is the same for compound events as for simple events: the theoretical probability is the fraction of the sample space in which the compound event occurs.

Core Standards · Students can and do:

a. Compute experimental probabilities from data sets, including data sets generated by simulations or repeated sampling experiments.
b. Compute experimental probabilities to estimate theoretical probabilities of compound events when no theoretical probability model is apparent.
c. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams; identify members of the sample space in which specified events occur.
d. Compute theoretical probabilities for compound events by counting members of the sample space.
e. Compare experimental probabilities to theoretical probabilities for multi-stage random experiments, examining the assumptions of the theoretical model when the two conflict.

Mathematics: High School—Expressions

A Coherent Understanding of Expressions

[Final draft of CCR narrative goes here.]

Seeing structure in expressions

Core Standards · Students understand that:

1. Different forms of expression for functions reveal different properties of the function; a purpose in transforming expressions is to find those properties.

   For example, factoring a quadratic expression reveals the zeros of the function it defines, and putting the expression in vertex form reveals the maximum or minimum of the function; the expression 1.15^t can be rewritten in the form $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate monthly interest rate if the annual rate is 15%.

2. The laws of exponents for whole number exponents follow from an understanding of exponents as indicating repeated multiplication, and from the associative property of multiplication.
3. The interpretation of zero, fractional and negative exponents follows from extending the laws of exponents to those values.

   For example, since $(x^{1/3})^3 = x^{(1/3)\cdot 3} = x^1 = x$, $x^{1/3}$ is the cube root of x.

4. Complex expressions can be interpreted by "chunking": temporarily viewing a part of the expression as a single entity.

Core Standards · Students can and do:

a. Factor, expand, and complete the square in quadratic expressions.
b. Use chunking to see expressions in different ways that suggest ways of rewriting them.

   For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.

c. Rewrite expressions using the laws of exponents.

   For example $(x^{1/2})^3 = x^{3/2}$ and $1/x = x^{-1}$.

d. Use the laws of exponents to interpret expressions for exponential functions, recognizing fractional exponents as indicating roots of the base and negative exponents as indicating the reciprocal of a power.

   For example, identify the relative rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.2)^{10t}$, $y = (1.01)^{t/12}$, and recognize that any non-zero number raised to the 0 power is 1 (for example, $12(1.05)^0 = 12$). Avoid common errors such as confusing as $6(1.05)^t$ with $(6\cdot 1.05)^t$ and $5(0.03)^t$ with $5(1.03)^t$.

e. Given an expression for an exponential function, identify whether it represents exponential growth or decay.
f. Using a method such as the factorization $(x^n–1) = (x–1)(x^{n-1} + \ldots + 1)$ where n is a whole number, prove the formula for the sum of a geometric series, and use the formula to solve problems.

   Include problems involving compound interest and mortgage payments.

The arithmetic of polynomials and rational functions

Core Standards · Students understand that:

1. Polynomials form a system analogous to the integers, closed under the operations of addition, subtraction, and multiplication.

2. A polynomial of degree n has n complex roots, where roots are counted according to multiplicity.
3. For a polynomial $p(x)$, $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
4. The Binomial Theorem gives the expansion of $(x + a)^n$ in powers of x for a whole number n and a real number a, with coefficients determined for example by Pascal's triangle. The Binomial Theorem can be proved by mathematical induction.
5. Rational functions are fractions whose numerator and denominator are polynomials, and the rational functions are closed under the operation of division by a nonzero rational function.

Core Standards · Students can and do:

a. Add, subtract and multiply polynomials.
b. Identify zeros of polynomials when suitable factorizations are available, and graph polynomials.
c. Transform simple rational functions using the Properties of Arithmetic and the rules for operations on fractions.
d. Identify zeros and asymptotes of rational functions, when suitable factorizations are available, and graph rational functions.
e. Divide polynomials by monomials

Mathematics: High School—Equations

A Coherent Understanding of Equations

[Final draft of CCR narrative goes here.]

Building equations to model relations between quantities

Core Standards · Students understand that:

1. Choosing a unit for a general quantity (e.g. length) establishes a correspondence between specific instances of the quantity (e.g. lengths of specific objects) and numbers called coordinates.
2. A relation between two quantities can be represented by an equation in variables representing coordinates for the quantities; by a graph on a pair of axes marked with units for the quantities; and by a table of coordinate pairs from the relation. The graph and the table show pairs that are solutions to the equation.

Core Standards · Students can and do:

a. Build equations to express relations between quantities and solve problems.

Include equations arising from situations involving linear, quadratic, simple rational, and exponential functions.

b. Rearrange formulas to isolate a quantity of interest.

c. Build systems of equations and solve problems involving systems of equations.

Reasoning with equations and inequalities

Core Standards · Students understand that:

1. To solve an equation algebraically, one assumes it is true and deduces the solutions, often in steps that replace it with a simpler equation whose solutions include the solutions of the original one.
2. Adding a number to both sides of an equation, or multiplying both sides by a nonzero number, leads to an equation that has exactly the same solutions as the original.
3. If the product of two numbers is zero, then at least one equals zero, and conversely. This principle is the basis for solving equations by factoring.
4. Multiplying both sides of an equation by an expression that can be zero for certain values of the variables in it, or squaring both sides of an equation, can lead to an equation that has more solutions than the original. Evaluating these in the original equation eliminates extraneous solutions.
5. The method of completing the square can transform any quadratic equation in x into an equivalent equation of the form $(x - p)^2 = q$. This leads to the quadratic formula.
6. Equations not solvable in one number system may have solutions in a larger number system.
7. Equations of the form $f(x) = g(x)$ can be solved graphically by finding the intersections (if any) of the graphs of $f(x)$ and $g(x)$.
8. The relationship between a function f and its inverse (if it has one) can be used to solve equations of the form $f(x) = c$. For example, a logarithmic function can help solve exponential equations, and an inverse trigonometric function can help solve trigonometric equations.
9. Given a system of linear equations, adding a m*u*ltip*l*e of one equation to another produces a system with the same sol*u*tio*n*s. This principle, combined with principles already encountered with equations in one variable, allows for the simplification of systems.

10. The solutions to an equation in two variables form a graph—a set of points, often a curve or a line, in the coordinate plane.
11. The solutions to two equations in two variables (if any) can be visualized as the points of intersection of their graphs, because those points satisfy both equations simultaneously.
12. The solution to a system of inequalities in two variables can be visualized as the intersection of the regions in the plane defined by the inequalities.

Core Standards · Students can and do:

a. Solve simple rational and radical equations, noting and explaining extraneous solutions.
b. Solve quadratic equations over the real numbers by completing the square, using the quadratic formula and factoring.
c. Solve linear inequalities in one variable and graph the solution set on a number line.

   Emphasize solving the associated equality and determining on which side of the solution of the associated equation the solutions to the inequality lie.

d. Solve linear systems of equations algebraically, focusing on pairs of linear equations in two variables.
e. Graph a system of two linear or quadratic equations in two unknowns and estimate the solution from a graph.
f. Graph the solution set of a linear inequality in two variables.
g. Use the properties of logarithms to solve equations involving exponential functions.
h. Use inverse trigonometric functions to solve equations of the form $A\sin(Bx + C) = D$.
i. Find complex roots of quadratic equations.
j. Solve a system of two quadratic equations in two unknowns.

Mathematics: High School—Functions

A Coherent Understanding of Functions

[Final draft of CCR narrative goes here.]

Interpreting functions

Core Standards · Students understand that:

1. The domain of a function is the set of its inputs, and the range is the set of its outputs.
2. Function notation uses a letter to stand for a function. If f is a function and x is a number in its domain, then $f(x)$ indicates the output of f corresponding to the input x.
3. Functions can be described by key characteristics, including: zeros; vertical intercept; extreme points; average rates of change (over intervals); intervals of increasing, decreasing and/or constant behavior; and end behavior.
4. Linear, quadratic and exponential functions are defined by expressions that have forms specific to each type, in which the parameters can often be interpreted in terms of characteristics of the graph.
5. An equation in two variables implicitly expresses one variable as a function of the other if there are no points on the graph having the same value of the first variable but different values of the second.
6. When x is a power of ten, the common logarithm $\log(x)$ tells the exponent. When x lies between 10^n and 10^{n+1}, $\log(x)$ lies between n and $n+1$.

Core Standards · Students can and do:

a. Describe qualitatively the functional relationship between two quantities by reading a graph; e.g., where the function is increasing or decreasing, what its long run behavior appears to be, and whether it appears to be periodic.
b. Sketch a graph that exhibits the qualitative features of a function that has been described verbally.
c. Compare values and properties of two functions represented in different ways (algebraically, graphically, numerically in tables, or by verbal descriptions).
d. Relate the domain and range of a function to its graph and, where applicable, to the quantitative relationship it describes.
e. Describe the qualitative behavior of common types of functions using graphs and tables.

   Identify: intercepts; intervals where the function is increasing, decreasing, positive or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. Use technology to explore the effects of parameter changes on the graphs of linear, power, quadratic, square root and cube root, polynomial, simple rational, exponential, logarithmic, sine and cosine, absolute value and step functions.

f. Interpret the parameters in the general expressions for linear, quadratic, and exponential functions, and draw conclusions about the parameters by inspection of the graph.
g. Given a function f, and given a constant c, evaluate $f(c)$ if possible and find solutions to $f(x) = c$ (if they exist). Where appropriate, relate the possibility of evaluation to the domain and the existence or nonexistence of solutions to the range.

Building functions

Core Standards · Students understand that:

1. Varying a parameter in the general expression for a linear, quadratic or exponential function can (often) be interpreted as performing a geometric transformation on the graph. This can be used to adjust a function to model a particular situation.

2. Composing a function *f* with a function *g* creates a new function called the composite function—for an input number x, the output of the composite function is $f(g(x))$.

3. The inverse of a function "undoes" what the function does; that is, composing the function with its inverse in either order returns the original input.

4. Sequences are functions whose domain is the whole numbers, and they can be defined recursively as well as explicitly. Arithmetic sequences are linear functions and geometric sequences are exponential functions.

Core Standards · Students can and do:

a. Make graphs of linear, quadratic, cubic, absolute value and exponential functions, and, given the graph of one of these types, identify the type.
b. Sketch graphs of quadratic functions presented in the form $y = ax^2+bx+c$, $y = a(x\text{-}h)^2+k$ and $y = a(x - p)(x - q)$ (without plotting points).
c. Solve problems involving quadratic functions, such as analyzing projectile motion and maximizing profit.
d. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $kf(x)$, $f(kx)$, and $f(x + k)$. Include both positive and negative k; find the value of k given the graphs.
e. Write an expression of the form $a(1+r)^t$ or ab^t for an exponential function to express a constant percent growth rate or a constant growth factor.
f. Evaluate composite functions and compose functions symbolically in simple cases (e.g. one or both functions linear).
g. Read values of an inverse function from a graph or a table, given that the function has an inverse.
h. For linear or simple exponential functions, find a formula for an inverse function by solving an equation.
i. For linear functions or simple exponential functions, verify symbolically by composition that one function is the inverse of another.
j. Write arithmetic and geometric sequences both recursively and in closed form, and translate between the two forms.

Linear vs. exponential behavior

Core Standards · Students understand that:

1. Linear functions grow by equal differences in equal time periods; exponential functions grow by equal factors in equal time periods.

2. The rate of change of a linear function is constant; the rate of change of an exponential function is proportional to the value of the function.

3. Exponential growth eventually outstrips polynomial growth (including, in particular, linear growth).

Students can and do:

a. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.
b. Interpret absolute and relative rates of change and use them to make predictions.
c. Identify the initial value and growth or decay rate from a table or graph of an exponential function.
d. Calculate and interpret the growth factor for an exponential function (presented symbolically or as a table) given a fixed time interval. Estimate the growth factor from a graph.
e. Recognize a quantitative relationship as linear or exponential from description of a situation.

Trigonometric functions

Core Standards · Students understand that:

1. The unit circle in the coordinate plane enables one to extend the domains of the sine, cosine and tangent functions of right-triangle trigonometry to the real numbers.
2. Trigonometric functions are periodic by definition, and sums and products of these functions are periodic.
3. Restricting trigonometric functions to a domain on which they are always increasing or always decreasing allows for the construction of an inverse function.

Core Standards · Students can and do:

a. Use radian measure and revisit graphs of trigonometric functions in terms of radians.
b. Use the unit circle to determine geometrically the values of sine, cosine, tangent for multiples of $\pi/4$ and $\pi/3$; commit sines and cosines of principal angles to memory.
c. Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
d. Solve simple trigonometric equations formally using inverse trigonometric functions;[64] evaluate solutions using technology.
e. Explain relationships between the identity $\sin^2 x + \cos^2 x = 1$, the equation of a circle, and the Pythagorean theorem.
f. Explain proofs of the sine and cosine addition and subtraction formulas.
g. Use trigonometric identities to simplify expressions.
h. Use trigonometric functions to solve problems in science, economics or other fields where periodic phenomena occur.

[64] Solving trigonometric equations by means of the quadratic formula is optional.

Mathematics: High School—Coordinates

A Coherent Understanding of Coordinates.

[Final draft of CCR narrative goes here.]

Expressing geometric properties with equations

Core Standards · Students understand that:

1. The graph of a linear equation is the straight line through any two of its solutions. Conversely, any line is the set of solutions to some linear equation.
2. Two lines with well-defined slopes are parallel if their slopes are equal and perpendicular if their slopes multiply to -1.
3. The equation of a circle can be found using its definition and the Pythagorean theorem.
4. Transforming the graph of an equation by reflecting in the axes, translating parallel to the axes, or applying a dilation to one of the axes correspond to substitutions in the equation.

   For example, reflection in the y axis corresponds to $(x,y) \rightarrow (-x,y)$, translation vertically down by three units corresponds to $(x,y) \rightarrow (x,y+3)$, and dilating by a factor of 2 parallel to the x-axis corresponds to $(x, y) \rightarrow (x/2, y)$.
5. An ellipse is obtained by stretching a circle, leading to an equation of the form $x^2/a^2 + y^2/b^2 = 1$.
6. The formula $A = \pi ab$ for the area of an ellipse can be derived from the formula for the area of a circle.

Core Standards · Students can and do:

a. Write the equation of a line in point-slope form, slope-intercept form, or standard form.
b. Identify parallel and perpendicular lines in a coordinate plane, and use the relationship between slopes of parallel and perpendicular lines to solve problems. Know the equations of vertical and horizontal lines.
c. Find the point on the segment between two given points that divides the segment in a given ratio.
d. Complete the square to find the center and radius of a circle given by an equation.
e. Find an equation for an ellipse given the lengths of its major and minor axes; calculate the area of an ellipse.
f. Use coordinates to solve geometric problems.

   Include proving simple geometric theorems algebraically, using coordinates to compute perimeters and areas for triangles and rectangles, finding midpoints of line segments, finding distances between pairs of points and determining when two lines are parallel or perpendicular.

Vectors and matrices[65]

Core Standards · Students understand that:

1. Vectors are quantities having both magnitude and direction. They are typically represented by directed line segments.
2. On a coordinate plane, vectors are determined by the coordinates of their initial and terminal points or by their x- and y-components.

[65] Limit to vectors in the plane and 2x2 matrices.

3. Vectors can be added end-to-end, component-wise, or by the parallelogram rule. The length of the sum of two vectors is typically not the sum of the lengths.
4. Translations of the plane can be represented by vectors.
5. Vectors are often used to describe "directed quantities" in physics, such as position, velocity, acceleration and force. Vector addition is used to find resultant forces or compute displacements.
6. Multiplying a 2x2 matrix into a vector produces another vector. This can be viewed as a transformation of the plane.
7. A system of two linear equations in two variables can be represented as a single matrix equation in a vector variable.
8. Matrices can be added, subtracted and multiplied.
9. The zero and identify matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a 2x2 matrix determines whether it has a multiplicative inverse.

Core Standards · Students can and do:

a. Represent vectors graphically.
b. Perform basic vector operations (addition, subtraction, scalar multiplication) both graphically and algebraically.
c. Use vectors to model and solve problems.
d. Use trigonometry to decompose a vector into perpendicular components.
e. Add, subtract and multiply matrices.
f. Represent systems of equations as matrix equations.
g. Find the inverse of a matrix if it exists and use it to solve equations.

Complex Numbers

Core Standards · Students understand that:

1. To solve quadratic equations that have no solutions in the real numbers, the number system can be extended to include the square roots of –1, creating a closed number system called the complex numbers.
2. The Properties of Arithmetic and the relation $i^2 = -1$ can be used to perform operations on complex numbers.
3. All polynomials can be factored over the complex numbers, e.g. as in $x^2 + 4 = (x + 2i)(x - 2i)$.
4. Complex numbers can be visualized on the complex plane. Real numbers fall on the horizontal (real) axis, and imaginary numbers fall on the vertical axis.
5. On the complex plane, arithmetic of complex numbers can be interpreted geometrically: addition is analogous to vector addition, and multiplication can be understood as rotation and dilation about the origin. Complex conjugation is reflection across the real axis.
6. The absolute value (or modulus) of a complex number is defined as its distance from the origin in the complex plane. On the complex plane, as on the real line, the distance between numbers is the absolute value of the difference, and the midpoint of a segment is the average of the numbers at its endpoints.
7. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ links complex numbers to trigonometry.

Core Standards · Students can and do:

a. Add, subtract and multiply complex numbers.
b. Find the conjugate of complex a number and use it to find absolute values and divide complex numbers.
c. Graph complex numbers in both rectangular and polar form and interpret arithmetic of complex numbers geometrically.

d. Solve quadratic equations over the complex numbers.
e. Convert complex numbers between rectangular and polar form.
f. Re-derive trigonometric identities using complex methods.

Mathematics: High School—Modeling

A Coherent Understanding of Modeling.

Modeling uses mathematics to help us make sense of the real world—to understand quantitative relationships, make predictions, and propose solutions.

A model can be very simple, such as a geometric shape to describe a physical object like a coin. Even so simple a model involves making choices. It is up to us whether to model the solid nature of the coin with a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. For some purposes, we might even choose to adjust the right circular cylinder to model more closely the way the coin deviates from the cylinder.

In any given situation, the model we devise depends on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models we can create and analyze is constrained as well by the limitations of our mathematical and technical skills. For example, modeling a physical object, a delivery route, a production schedule, or a comparison of loan amortizations each requires different sets of tools. Networks, spreadsheets and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations. One of the insights provided by mathematical modeling is that essentially the same mathematical structure might model seemingly different situations.

The basic modeling cycle is one of (1) identifying the key features of a situation, (2) creating geometric, algebraic or statistical objects that describe key features of the situation, (3) analyzing and performing operations on these objects to draw conclusions and (4) interpreting the results of the mathematics in terms of the original situation. Choices and assumptions are present throughout this cycle.

The modeling cycle and general tools

Core Standards · Students understand that:

1. The behavior of quantities in physical, economic, public policy, social and everyday situations can be modeled using mathematics. Mathematics is used to model relationships among quantities, constraints and objectives.
2. Models are formulated to answer questions about the world based on an analysis of the situation and a conceptual model that involves assumptions and choices.
3. Quantities in the situation are represented by variables in the model, usually through measurement. Modeling includes decisions about what to measure and how, and how well the measurements work for the purpose.
4. Mathematical knowledge and skill are required in order to get results from a mathematical model—even to devise a model in the first place. Areas of mathematics commonly used in modeling include linear, quadratic, exponential or other functions; probability and statistics; and geometry (solid, plane and coordinate). In active modeling, fluency with math content is required in order to focus on the larger problem.
5. Technology is often required in order to obtain results from a model.
6. The results of a mathematical model must be evaluated against evidence and the phenomena at hand. If the mathematics is correct, then unreasonable results point to unreasonable assumptions and the need to revise the model.
7. Real-world problems do not announce that they are amenable to mathematical analysis and solution; bringing mathematics to bear on such a problem is a highly creative act.

Core Standards · Students can and do:

a. Creatively apply the mathematics they know to situations in which it only imperfectly applies—and achieve useful results by doing so.

For example, independently choose to describe HIV transmission as a random event with a fixed probability per sexual encounter.

b. Decide what measures are relevant to a problem.

For example, given the purpose at hand, is traffic safety best measured in terms of fatalities per year or fatalities per vehicle-mile? (or fatalities per 100 million vehicle-miles?)

c. Use network diagrams or other techniques to visualize complex situations with many factors, causes or agents.

For example, what agents and factors are responsible for setting the price of gasoline? How do they interact?

d. In situations with many factors, causes or agents, organize the factors/causes/agents into a hierarchy of importance.

For example, what are the primary, secondary, and relatively rare causes of lung cancer?

e. Use order of magnitude estimates, unprompted, to identify important effects, disregard unimportant effects and predict results of a more detailed model.
f. Use 2-by-2 tables, flowcharts, and other strategies to organize information and manage scenarios.

Modeling with geometry, equations, functions, probability, and statistics

Core Standards · Students can and do:

a. Model physical objects with geometric shapes.

Include common objects that can reasonably be idealized as two- and three-dimensional geometric shapes. Identify the ways in which the actual shape varies from the idealized geometric model.

b. Model situations with equations and inequalities.

Include situations well described by a linear inequality in two variables or a system of linear inequalities defining a region in the plane.

c. Model situations with common functions.

Include situations well described by linear, quadratic or exponential functions; and situations that can be well described by inverse variation ($f(x) = k/x$). Include identifying a family of functions that models features of a problem, and identifying a particular function of that family and adjusting it to fit by changing parameters. Understand the recursive nature of situations modeled by linear and exponential functions.

d. Model situations using probability and statistics.

Include using simulations to model probabilistic situations; describing the shape of a distribution of values and summarizing a distribution with measures of center and variability; modeling a bivariate relationship using a trend line or a regression line.

Mathematics: High School—Statistics

A Coherent Understanding of Statistics.

[Final draft of CCR narrative goes here.]

Summarizing and interpreting categorical, count and measurement data

Core Standards · Students understand that:

1. Statistical methods take variability into account to support making informed decisions based on quantitative studies designed to answer specific questions.
2. Visual displays and summary statistics condense the information in data sets into usable knowledge.

Core Standards · Students can and do:

a. Summarize comparative or bivariate categorical data in two-way frequency tables; interpret joint, marginal and conditional relative frequencies in the context of the data.
b. Compare data on two or more count or measurement variables by using plots on the real number line (dot plots, histograms and box plots); use appropriate statistics to summarize center (median, mean) and spread (interquartile range, standard deviation) of the data sets; interpret changes in shape, center and spread in the context of the data sets, accounting for possible effects of extreme data points.
c. Summarize bivariate quantitative data by giving a regression line and a measure of goodness of fit.

Making inferences and justifying conclusions drawn from data

Core Standards · Students understand that:

1. Statistics is a process for making inferences about population parameters based on a sample from that population; randomness is the foundation for statistical inference.
2. The design of an experiment or sample survey is of critical importance to analyzing the data and drawing conclusions.

Core Standards · Students can and do:

a. Use probabilistic reasoning to decide if a specified model is consistent with a given data-generating process.
b. Recognize the purposes of and differences among sample surveys, experiments and observational studies; explain how randomization relates to each.
c. Use data from a sample survey to estimate a population parameter.
d. Use data from a randomized experiment to compare two treatments.
e. Evaluate reports based on data.

Mathematics: High School—Probability

A Coherent Understanding of Probability.

[Final draft of CCR narrative goes here.]

Modeling random events with finite sample spaces

Core Standards · Students understand that:

1. Random phenomena can be modeled mathematically using a sample space in which sample points represent distinct outcomes, and in which each sample point is assumed to have the same probability.
2. Events are subsets of a sample space that can be defined using characteristics (or categories) of the sample points, as well as unions, intersections, or complements thereof ('and', 'or', 'not'). A sample point may belong to several events (categories).
3. If A and B are two events (categories), then the conditional probability of A given B, denoted by $p(A \mid B)$, is the fraction of sample points in B that also lie in A.
4. The laws of probability can be used to generate new probabilities from known probabilities.

Core Standards · Students can and do:

a. Compute theoretical probabilities of compound events by constructing and analyzing representations, including tree diagrams, systematic lists, and Venn diagrams
b. Use the addition and multiplication laws of probability to compute probabilities of complementary, disjunctive, and compound events.
c. Apply concepts such as intersections, unions and complements of events, and conditional probability and independence, to define or analyze compound events, calculate probabilities, and solve problems.
d. Construct and interpret two way tables to show probabilities when two characteristics (or categories) are associated with each sample point. Use a two way table to determine conditional probabilities.
e. Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations.
f. Use permutations and combinations to compute probabilities of compound events and solve problems.

Experimenting and simulating to model probabilities

Core Standards · Students understand that:

1. Sets of data obtained from surveys, simulations, or other means, can be used as probability models, by treating the data set itself as a sample space, in which the sample points are the individual pieces of data. The probability of an event within the data set is its relative frequency.
2. The law of large numbers provides the basis for estimating certain probabilities by use of empirical relative frequencies.
3. The probability of an outcome can be interpreted as an assertion about the long-run proportion of the time the outcome will occur if the random experiment is repeated a large number of times. The observed proportion of occurrence for the outcome of interest can be used as an estimate of the relevant probability.

Core Standards · Students can and do:

a. Calculate experimental probabilities by performing simulations or experiments involving a probability model and using relative frequencies of outcomes.

b. Compare the results of simulations (e.g., random number tables, random functions, and area models) with predicted probabilities. When there are substantial discrepancies between predicted and observed probabilities, explain them in terms of the assumptions of the probability model.
c. Use the relationship between conditional probabilities and relative frequencies in contingency tables to analyze decision problems.
d. Use the mean and standard deviation of a data set to fit it to a normal distribution (bell-shaped curve) and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets and tables to estimate areas under the normal curve.
e. Apply the binomial theorem to solve probability problems.

Using probability to make decisions

Core Standards · Students understand that:

1. A probability distribution is a collection of probabilities $\{p_1, \ldots, p_n\}$ for a set of mutually exclusive and jointly exhaustive events $\{E_1, \ldots, E_n\}$. The probabilities in a probability distribution sum to unity.
2. A random variable attaches a value to each event in a probability distribution. The expected value of the random variable is the weighted average of its possible values, with weights given by their respective probabilities.
3. When the possible outcomes of a decision can be assigned probabilities and payoff values, the decision can be analyzed as a random variable with an expected value, e.g. of a wager. If possible, this is the first thing to compute in a decision context.

Core Standards · Students can and do:

a. Calculate expected value to analyze mathematical fairness, payoff.
b. Evaluate and compare options in situations where all of the available options share the same expected value but carry different levels of risk.
c. Analyze each of two options and make a quantitatively informed decision in situations where one option has both a higher expected return and a higher level of risk. Include both low-stakes and high-stakes decisions.
d. Analyze decision problems using probability concepts.

Mathematics: High School—Geometry

A Coherent Understanding of Geometry.

[Final draft of CCR narrative goes here.]

Triangle Congruence

Core Standards · Students understand that:

1. Rigid motions move lines to lines and segments to segments; preserve the distance between points; and preserve measures of angles.
2. Two geometric figures are congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition.
3. Criteria for triangle congruence can be thought of as answers to the following question: What information about the measures in a triangle ensures that all triangles drawn with those measures are congruent?
4. Criteria for triangle congruence can be established using rigid motions.

Core Standards · Students can and do:

a. Use (in reasoning and problem solving) precise definitions of angles, polygons, parallel and perpendicular lines, rigid motions (rotations, reflections, translations), parallelograms and rectangles; commit these definitions to memory.

b. Prove theorems about lines and angles; test conjectures and identify logical errors in fallacious proofs.

   Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; measures of supplementary angles sum to 180°; two lines parallel to a third are parallel to each other; points on a perpendicular bisector of a segment are exactly those equidistant from the segment's endpoints.

c. Prove theorems about triangles; test conjectures and identify logical errors in fallacious proofs.

   Theorems include: measures of interior angles of a triangle sum to 180°, base angles of isosceles triangles are equal, the triangle inequality, the longest side of a triangle faces the largest side and vice-versa, the exterior-angle inequality, and the segment joining midpoints of two sides of a triangle parallel to the third side and half the length.

d. Use and prove properties of and relationships among special quadrilaterals: parallelogram, rectangle, rhombus, square, trapezoid and kite.

e. Characterize parallelograms in terms of equality of opposite sides, in terms of equality of opposite angles, and in terms of bisection of diagonals; characterize rectangles as parallelograms with equal diagonals.

Similarity, Right Triangles and Trigonometry

Core Standards · Students understand that:

1. The dilation of a given line is parallel to the given line. (In particular, lines passing through the center remain unchanged.)
2. The dilation of a given segment is parallel to the given segment and longer or shorter in the ratio given by the scale factor. A dilation leaves a segment unchanged if and only if the scale factor is 1.
3. The assumed properties of dilations can be used to establish the AA, SAS and SSS criteria for similarity of triangles.
4. Similarity allows one to view side ratios in right triangles as properties of the angles themselves, leading to elementary definitions of sine, cosine and tangent.

Core Standards · Students can and do:

a. Use triangle similarity criteria to solve problems and to prove relationships in geometric figures.
b. Prove that two lines with well-defined slopes are parallel if and only if they have the same slope, and perpendicular if and only if the product of their slopes is equal to –1.
c. Give an informal explanation using successive approximation that a dilation of scale factor r changes the length of a curve by a factor of r and the area of a region by a factor of r^2.
d. Use and explain the relationship between the trigonometric ratios of complementary angles.
e. Use trigonometric ratios and the Pythagorean theorem to solve right triangles[66] in applied problems.

Circles

Core Standards · Students understand that:

1. All circles are similar.
2. There is a unique circle through three non-collinear points, or tangent to three non-concurrent lines.

Core Standards · Students can and do:

a. Identify and describe relationships among angles, radii, and chords.

   Include the relationship between central, inscribed and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

b. Identify and define radius, diameter, chord, tangent, secant and circumference.
c. Determine the arc lengths and the areas of sectors of circles, using proportions.
d. Construct a tangent line from a point outside a given circle to the circle.
e. Prove and use basic theorems about circles, and use these theorems to solve problems. Include:
   - Symmetries of a circle
   - Similarity of a circle to any other
   - Tangent line, perpendicularity to a radius
   - Inscribed angles in a circle, relationship to central angles, and equality of inscribed angles
   - Properties of chords, tangents and secants as an application of triangle similarity.

Axiomatic Systems

Core Standards · Students understand that:

1. Mathematical statements are proven or disproven by deductive reasoning. Conjectures can arise from inductive reasoning, but they cannot be proven that way.
2. Precise definitions make possible rigorous logical reasoning, and definitions shared in common make possible the objective evaluation of one's own reasoning by others.
3. Logical reasoning requires avoiding common fallacies, such as using an example to prove the rule or confusing a statement with its converse.
4. Axiomatic systems require precise definitions, but some terms must be left "undefined." The axioms specify how the undefined terms behave.
5. The first three postulates of the *Elements* are models of straightedge and compass construction.
6. Hilbert and other mathematicians improved on the *Elements* by identifying its hidden assumptions and making them explicit with additional axioms.

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[66] A right triangle has five parameters, its three lengths and two acute angles. Given a length and any other parameter, "solving a right triangle" means finding the remaining three parameters. (It is worth reflecting on why this problem is well-posed.)

7. Three classical construction problems (trisecting an angle, duplicating a cube and squaring a circle) inspired the development of much important mathematics.
8. The Parallel Postulate (axiom) distinguishes Euclidean geometry from other geometries. Other geometries, such as spherical and hyperbolic geometry, use alternatives to the Parallel Postulate. Many theorems of Euclidean geometry are not theorems in other geometries.

Core Standards · Students can and do:

a. Use the terms point, line and plane to define other geometric terms as line segments, angles and rays.
b. With ruler and compass:
   - Divide a segment into any number of equal parts.
   - Given two segments of lengths r and s, construct a segment of length rs and one of length r/s.
   - Given a segment of length r, construct a segment of length $\sqrt{r}$.

Trigonometry of General Triangles

Core Standards · Students understand that:

1. The Law of Sines generalizes the side-angle inequality.
2. The Law of Cosines generalizes the Pythagorean theorem.
3. The Laws of Sines and Cosines embody the triangle congruence criteria, in that three pieces of information are usually sufficient to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that "Side-Side-Angle" is not a congruence criterion.

Core Standards · Students can and do:

a. Explain proofs of the Law of Sines and the Law of Cosines.
b. Use the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Geometric Measurement and Dimension

Core Standards · Students understand that:

1. Congruence plays a fundamental role defining the concepts of length, area and volume.
2. Areas of polygons can be computed by dissecting them into triangles and using the fundamental property of area, that the area of a dissected figure is the sum of the areas of its components.
3. Lengths of curves and areas of curved regions can be defined informally using the concept of "limit."
4. Cavalieri's principle allows one to understand volume formulas informally by visualizing volumes as stacks of thin slices.

Core Standards · Students can and do:

a. Give definitions of rectangular prism, (right) pyramid, (right circular) cone, (right circular) cylinder and sphere.

b. For a pyramid or a cone, give an heuristic argument to show why its volume is (1/3) its height times the area of its base.

c. Use the behavior of length and area under dilations to prove the formulas for the circumference and area of a circle.

d. Apply formulas and solve problems involving volume and surface area of right prisms, right circular cylinders and right pyramids, cones, spheres and composite figures.

e. Identify and apply the 3:2:1 relationship among volumes of circular cylinders, hemispheres and cones with same height and circular base and 3:1 relationship between volume of a prism and pyramid with same base area and height.

f. Identify cross-sectional shapes of slices of three-dimensional objects, and identify three-dimensional objects traced out by rotations of two-dimensional objects.

Mathematics: High School—Calculus

Calculus is an important part of the high school curriculum for a large and growing number of students. To see well-established standards for this course, please see course descriptions such as those of the College Board, International Baccalaureate Organization, or any of the following states: California, Florida, Hawaii, Indiana, Mississippi, Pennsylvania, South Carolina, Tennessee, Utah, and Virginia. We invite feedback from states as to whether they would like to see Calculus in future drafts of the Common Core Standards.

Progressions in Grades K–8

Note, a progression may appear in more than one band.

Grades K-5

Number

Counting and Cardinality
Base Ten Computation
Early Relations and Operations
Quantity and Measurement
Operations and the Problems They Solve
Fractions

Geometry

Shapes
Coordinates
Geometry

Data

Statistics

Grades 6-8

Number

The Number System

Algebra

Ratios and Proportional Relationships
Expressions and Equations
Functions and the Situations They Model

Geometry

Geometry

Data

Statistics
Probability

List of Progressions and Grade Ranges

Strand	Progression	Start	End
Number	Counting and Cardinality	K	K
	Early Relations and Operations	K	1
	Base Ten Computation	K	5
	Quantity and Measurement	K	5
	Operations and the Problems They Solve	2	4
	Fractions	3	5
	Ratios and Proportional Relationships	6	7
	The Number System	6	8
Geometry	Shapes	K	4
	Coordinate Geometry	5	5
	Geometry	6	8
Algebra*	Expressions and Equations	6	7
	Functions & The Situations They Model	8	8
Data*	Statistics	5	8
	Probability	7	8

* The Algebra and Data strands have concepts and skills in earlier grades that progressions in the Number strand.

GLOSSARY

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: $^3/4$ and $–^3/4$ are additive inverses of one another because $^3/4 + (–^3/4) = (–^3/4) + {}^3/4 = 0$.

Algorithm. A step by step routine that always gives some answer, rather than ever giving no answer; that always gives the right answer, and never gives a wrong answer; that can always be completed in a finite number of steps, rather than in an infinite number of steps; and that applies to all problems of a given type (e.g., adding any two multidigit whole numbers, or bisecting any angle). Cf. Wikipedia's "effective procedure," from which this definition is adapted.

Common logarithm. The common logarithm of x is the power to which you raise 10 in order to get x.

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by a sequence of rigid motions (rotations, reflections, and translations).

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center, and multiplies distances from the center by a common scale factor.

Integer. A positive whole number, a negative whole number, or 0.

Mean. The sum of the values in a list divided by the number of values in the list. (To be more precise, this defines the *arithmetic mean*.)

Median. In a list of values, the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: $^3/4$ and $^4/3$ are multiplicative inverses of one another because $^3/4 \times {}^4/3 = {}^4/3 \times {}^3/4 = 1$.

Range. The difference between the greatest and smallest values in a list of numbers.

Rational number. A number expressible in the form $^a/_b$ for integers a and $b \neq 0$. The rational numbers include positive and negative integers, positive and negative fractions, and 0.

Related fractions. Two fractions are *related* if one denominator is a factor of the other. (See Ginsburg, Leinwand and Decker (2009), *Informing Grades 1-6 Mathematics Standards Development: What Can Be Learned from High-Performing Hong Kong, Korea, and Singapore?*, Table A1, p. A-5, grades 3 and 4.)

Similarity transformation. A rigid motion followed by a dilation.

Single-place number. The numbers that result when a whole number between 1 and 9 (inclusive) is multiplied by the numbers 10, 100, 1000, etc.

Teen number. A whole number that is greater than or equal to 11 and less than or equal to 19.

Transitive property of measurement order. If one object is bigger than a second, and the second object is bigger than a third object, then the first object is bigger than the third object.